Singularity-free two-body equation with confining interactions in momentum space

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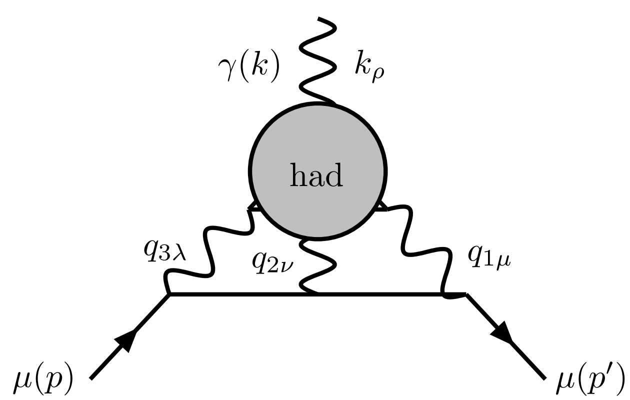
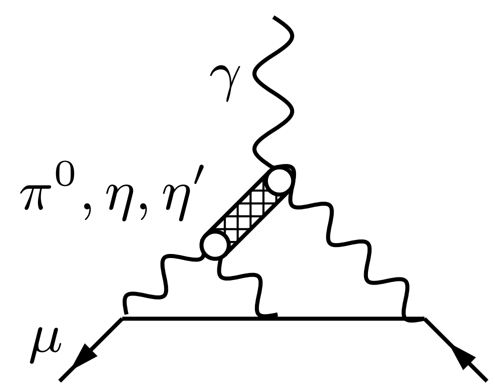
New challenges in meson physics

Motivation

- Upcoming intense experimental activity to explore meson structure
  GlueX (Jlab), PANDA (GSI)
- Search for exotic mesons (hybrids, glueballs)
- Need to understand "conventional" mesons in more detail
- Pion transition form factors
  Hadronic contributions to light-by-light scattering
  Source of uncertainty in anomalous magnetic moment of the muon
  Important in search for physics beyond the Standard Model





A unified model for all $q\bar{q}$ mesons

Much important work was done on meson structure

- Cornell-type potential models (Isgur and Godfrey, Spence and Vary, etc.)
  But: nonrelativistic (or "relativized"); structure of constituent quark and relation to existence of zero-mass pion in chiral limit not addressed
- Dyson-Schwinger approach (C. Roberts et al.)
  But: Euclidean space; only Lorentz vector confining interaction
- Lattice QCD (also Euclidean space), EFT, Bethe-Salpeter, Light-front, Point-form, ...

Our objectives

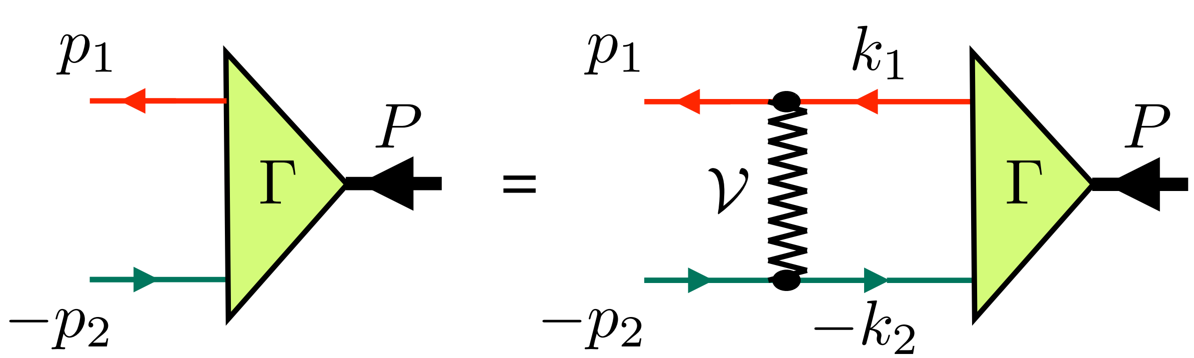
- Construct a model to describe all $q\bar{q}$-type mesons
- Covariant framework (CST) - light quarks require relativistic treatment
  Work in Minkowski space (physical momenta)
  Improve on previous work by Gross, Milana, Savkli
- Quark self-energy from $q\bar{q}$ interaction kernel (consistent quark mass function)
- Chiral symmetry: massless pion in chiral limit of vanishing bare quark mass
- Calculate meson spectrum and bound state vertex functions (wave functions)
- Pion elastic and transition form factors
- Learn about confining interaction (scalar vs. vector, etc.)

Covariant two-body bound-state equation

Start from the Bethe-Salpeter (BS) equation

$$\Gamma_{\rm BS}(p,P) = \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathcal{V}(p,k;P)\,S_1(k_1)\,\Gamma_{\rm BS}(k,P)\,S_2(k_2)$$

$$S_i(k_i) = \frac{1}{m_{0i} - \not{k}_i + \Sigma_i(\not{k}_i) - \mathrm{i}\epsilon} \qquad \Sigma_i(\not{k}_i) = A_i(k_i^2) + \not{k}_i B_i(k_i^2)$$



$P = k_1 - k_2$	total momentum
$k = (k_1 + k_2)/2$	relative momentum
$\Gamma(k,P)$ or $\Gamma(k_1,k_2)$	vertex function
$\mathcal{V}(p,k)$	kernel

Kernel contains confining interaction + color Coulomb (or constant)

In the BS equation it is effectively iterated to all orders

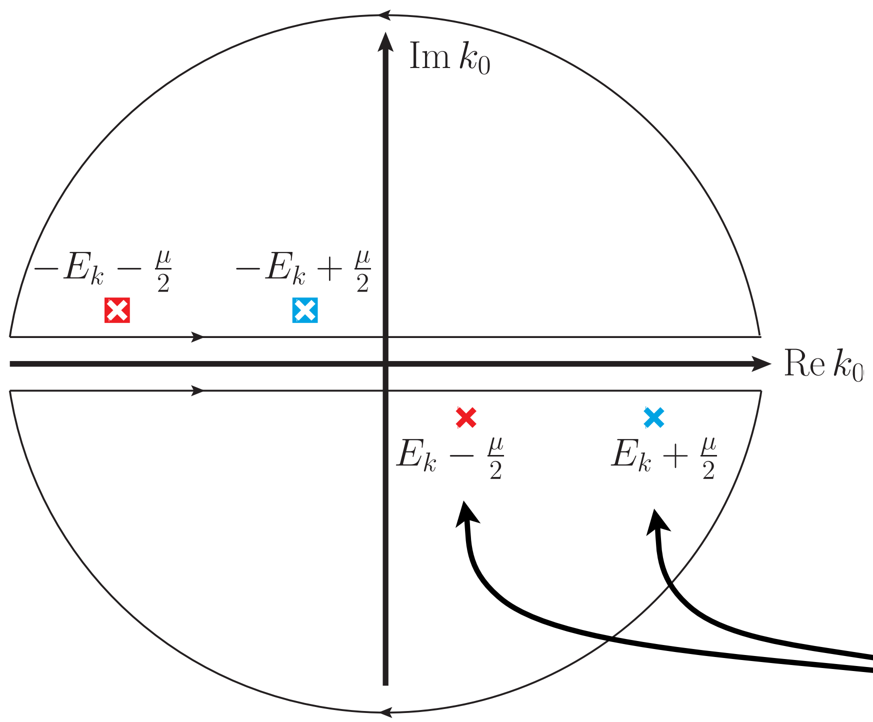
But the complete kernel is a sum of an infinite number of irreducible diagrams
→ has to be truncated (most often: ladder approximation)

Now we take a closer look at the loop integration over k_0

From Bethe-Salpeter to CST

Covariant Spectator Theory (CST)

Mini-review: A.S., F. Gross, Few-Body Syst. **49**, 91 (2010)



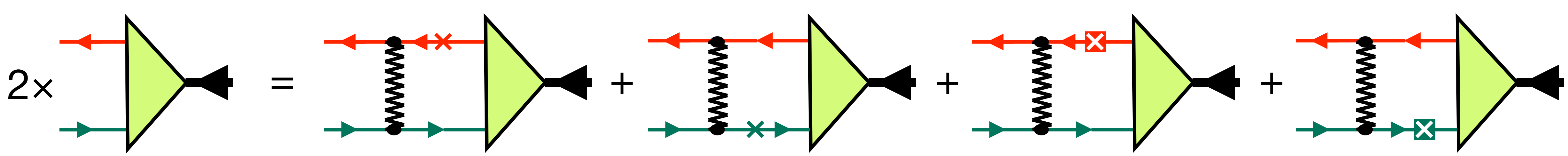
Integration over relative energy k_0:

- Keep only pole contributions from propagators
- Move kernel poles to higher-order kernels
- Cancellations between ladder and crossed ladder diagrams can occur
- Reduction to 3D loop integrations, but covariant
- Works very well in few-nucleon systems

If bound-state mass μ is small:
both poles are close together (both important)

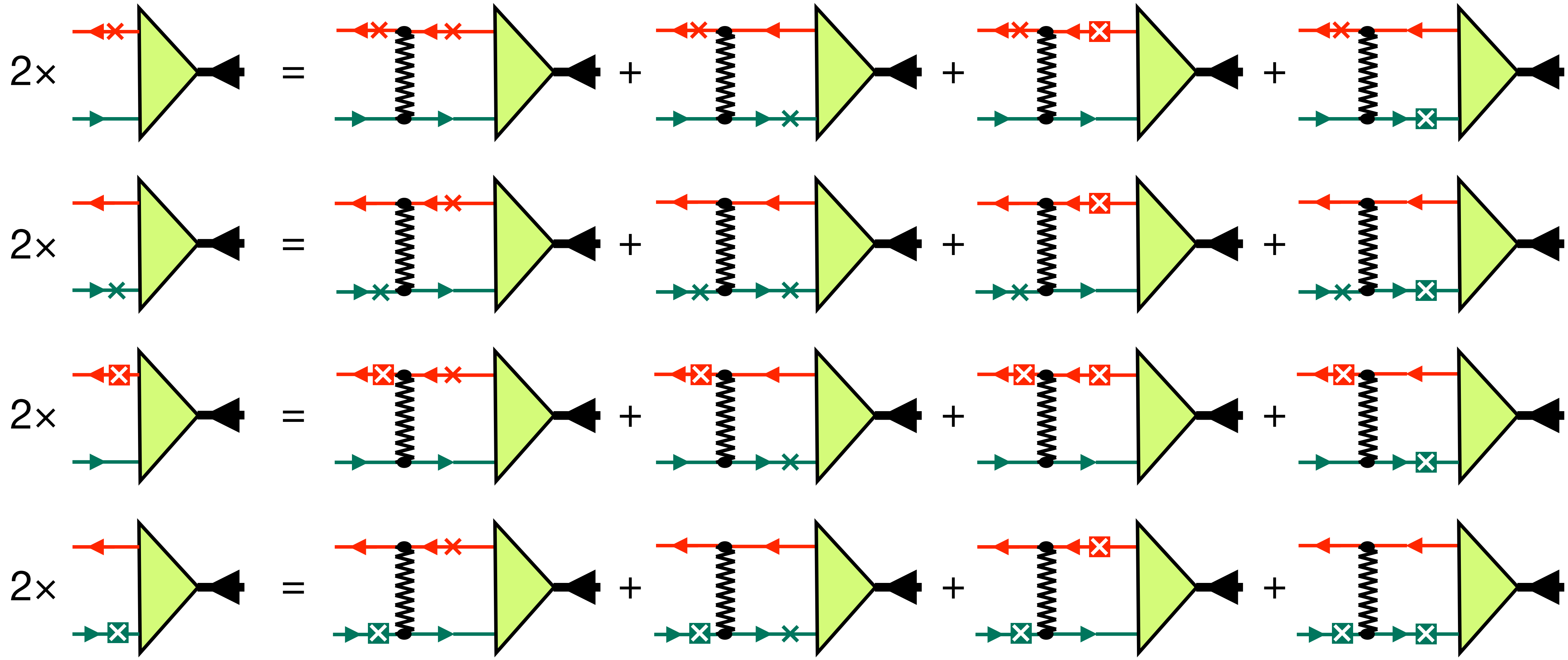
Symmetrize pole contributions from both half planes:
resulting equation is symmetric under charge conjugation

BS amplitude (approx.) CST amplitudes



Four-channel CST equation

Closed set of equations when external legs are systematically placed onshell



Approximations can be made for special cases

- Heavy-light quark systems: 2 channels
- Large bound-state mass: 1 channel

Nonrelativistic limit: Schrödinger equation

Quark self-energy, chiral symmetry, etc. → talk by E. Biernat

Linear confinement potential in momentum space

Model confining kernel as relativistic generalization of a nonrelativistic linear potential

But how to define a linear potential in momentum space?

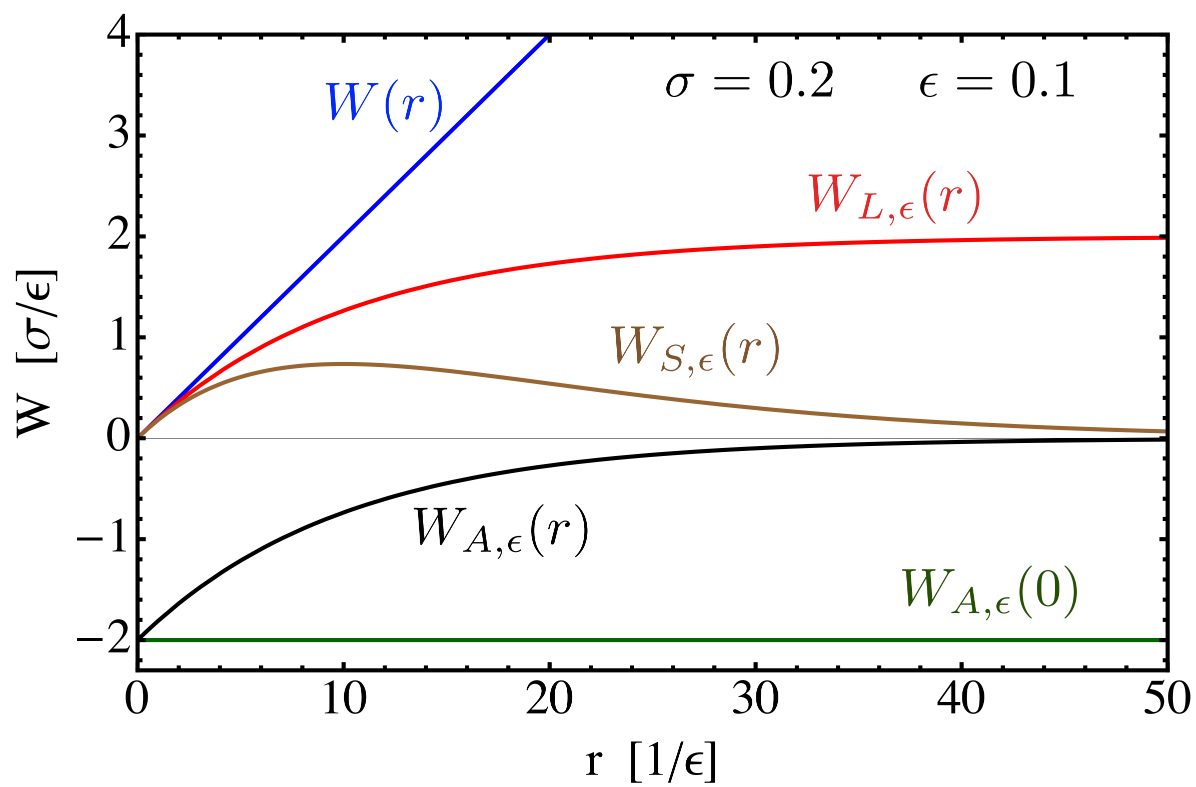
A possibility: $W(\mathbf{r}) = \sigma r \xrightarrow{\text{Screening}} W_{S,\epsilon}(\mathbf{r}) = \sigma r e^{-\epsilon r} = \sigma \frac{\partial^2}{\partial \epsilon^2} \frac{e^{-\epsilon r}}{r}$

→ use 2nd derivative of Fourier transform of Yukawa potential

More convenient: $W_{L,\epsilon}(\mathbf{r}) = \frac{\sigma}{\epsilon}\left(1 - e^{-\epsilon r}\right)$

with $\lim_{\epsilon \to 0} W_{L,\epsilon}(\mathbf{r}) = \sigma r$

Gross, Milana, PRD **43**, 2401 (1991)
Savkli, Gross, PRC **63**, 035208 (2001)



Useful to write it as

$$W_{L,\epsilon}(\mathbf{r}) = W_{A,\epsilon}(\mathbf{r}) - W_{A,\epsilon}(0)$$

where $W_{A,\epsilon}(\mathbf{r}) = -\frac{\sigma}{\epsilon} e^{-\epsilon r}$

Linear confinement potential in momentum space

Now calculate the Fourier transform of $W_{L,\epsilon}(\mathbf{r}) = W_{A,\epsilon}(\mathbf{r}) - W_{A,\epsilon}(\mathbf{0}) = \frac{\sigma}{\epsilon}\left(1 - e^{-\epsilon r}\right)$

$$V_{A,\epsilon}(\mathbf{q}) = \int d^3r\, e^{i\mathbf{q}\cdot\mathbf{r}} W_{A,\epsilon}(\mathbf{r}) = -\frac{8\pi\sigma}{(q^2+\epsilon^2)^2}$$

We can write the complete screened linear potential as

$$\boxed{V_{L,\epsilon}(\mathbf{q}) = V_{A,\epsilon}(\mathbf{q}) - (2\pi)^3\delta^{(3)}(\mathbf{q}) \int \frac{d^3q'}{(2\pi)^3} V_{A,\epsilon}(\mathbf{q}')}$$

$$\int \frac{d^3q}{(2\pi)^3} V_{A,\epsilon}(\mathbf{q}) = -\frac{\sigma}{\epsilon}$$

Inserting this potential into the momentum-space Schrödinger equation yields

$$\frac{p^2}{2m_R}\psi(\boldsymbol{p}) - 8\pi\sigma \int \frac{d^3k}{(2\pi)^3} \frac{\psi(\boldsymbol{k}) - \psi(\boldsymbol{p})}{[(\boldsymbol{k}-\boldsymbol{p})^2 + \epsilon^2]^2} = E\psi(\boldsymbol{p})$$

has a “built-in subtraction”

kernel becomes singular when $\epsilon \to 0$

Unscreened limit:

$V_L(\mathbf{q}) = \lim_{\epsilon\to 0} V_{L,\epsilon}(\mathbf{q})$

$$\boxed{\frac{p^2}{2m_R}\psi(\boldsymbol{p}) - 8\pi\sigma\, \mathrm{P}\!\int \frac{d^3k}{(2\pi)^3} \frac{\psi(\boldsymbol{k}) - \psi(\boldsymbol{p})}{(\boldsymbol{k}-\boldsymbol{p})^4} = E\psi(\boldsymbol{p})}$$

Cauchy principal value integration

Relativistic generalization: $\boldsymbol{q}^2 \to -q^2$

$$\boxed{V_A^{\mathrm{rel}}(q) = -\frac{8\pi\sigma}{q^4}}$$

4-vector q

The singularity in the CST equation is of the same type as in the nonrelativistic case

⟶ use nonrelativistic limit to test our numerical methods

S-waves

Exact solutions are known in r-space (Airy functions) ⟶ ideal test case!

S-wave linear potential

$$\langle p\,00|V_A|k\,00\rangle = 2\pi(-8\pi\sigma)\frac{2}{(p^2-k^2)^2}$$

Momentum-space Schrödinger equation with singular kernel

$$\frac{p^2}{2m_R}\psi_0(p) - \frac{4\sigma}{\pi}\mathrm{P}\int_0^\infty dk\,k^2\,\frac{\psi_0(k)-\psi_0(p)}{(p^2-k^2)^2} = E\psi_0(p)$$

This equation can be solved numerically, but the Cauchy principal value singularity makes it tedious!

Spence, Vary, PRD **35**, 2191 (1987)
Gross, Milana, PRD **43**, 2401 (1991)
Maung, Kahana, Norbury, PRD **47**, 1182 (1993)

But the singularity can be eliminated:

Analyzing the behavior of the integrand around the singular point $k{=}p$ one can apply a (second) subtraction, using $\mathrm{P}\int_0^\infty \frac{dk}{k^2-p^2} = 0$

$$\frac{p^2}{2m_R}\psi_0(p) - \frac{2\sigma}{\pi}\int_0^\infty \frac{dk}{k^2-p^2}\left[\frac{2k^2}{k^2-p^2}\Big(\psi_0(k)-\psi_0(p)\Big) - p\psi_0'(p)\right] = E\psi_0(p)$$

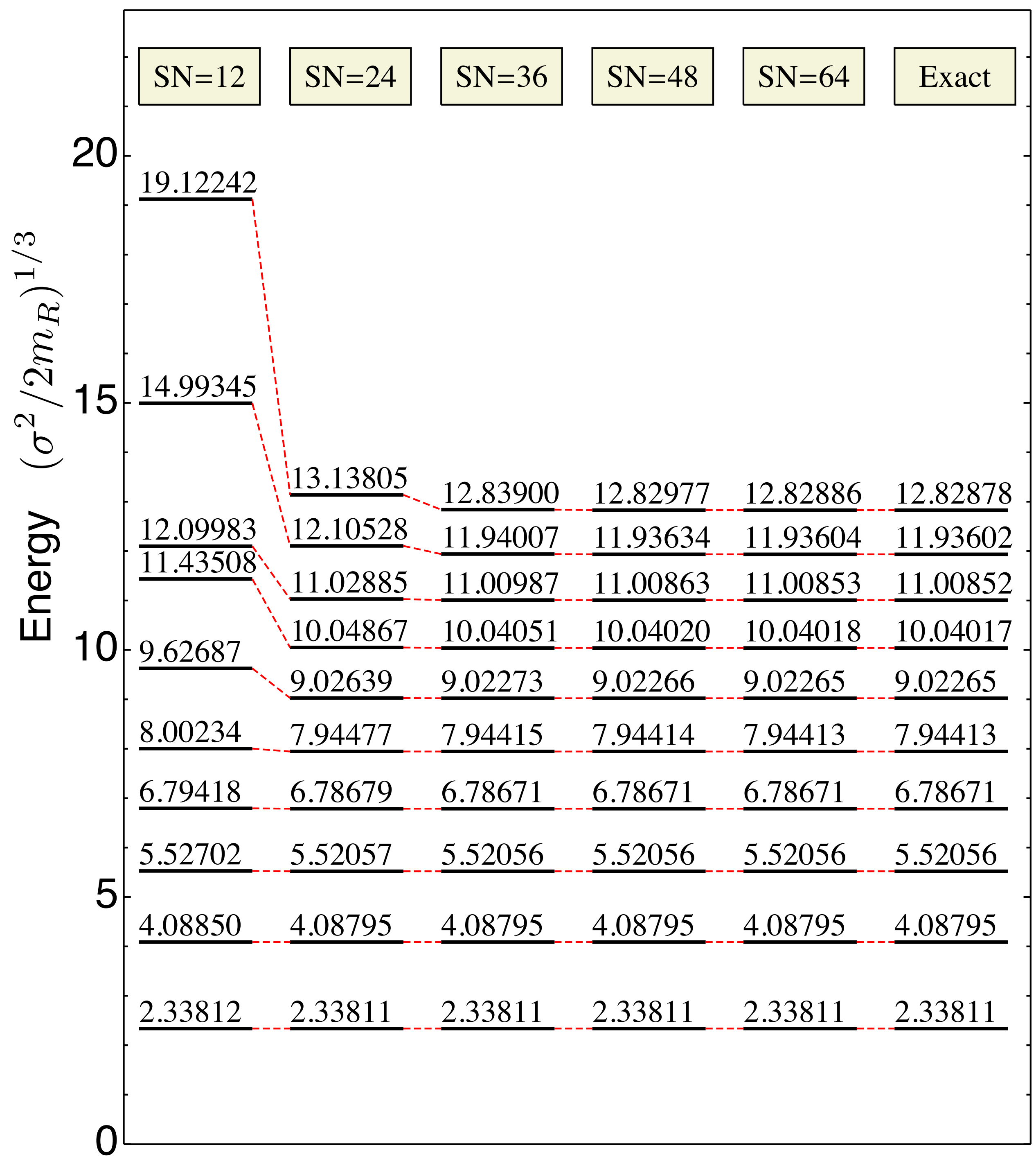
integrand now regular at $k{=}p$

The singularity-free equation is much easier to solve numerically and yields stable results (the derivative term does not lead to complications)

Numerical convergence

Our method: expand $\psi_0(p)$ in a basis of SN B-splines

Convergence with number of splines (ℓ=0)



Generalized eigenvalue problem for SN×SN matrices

Exact bound-state energies calculated from the roots of the Airy function

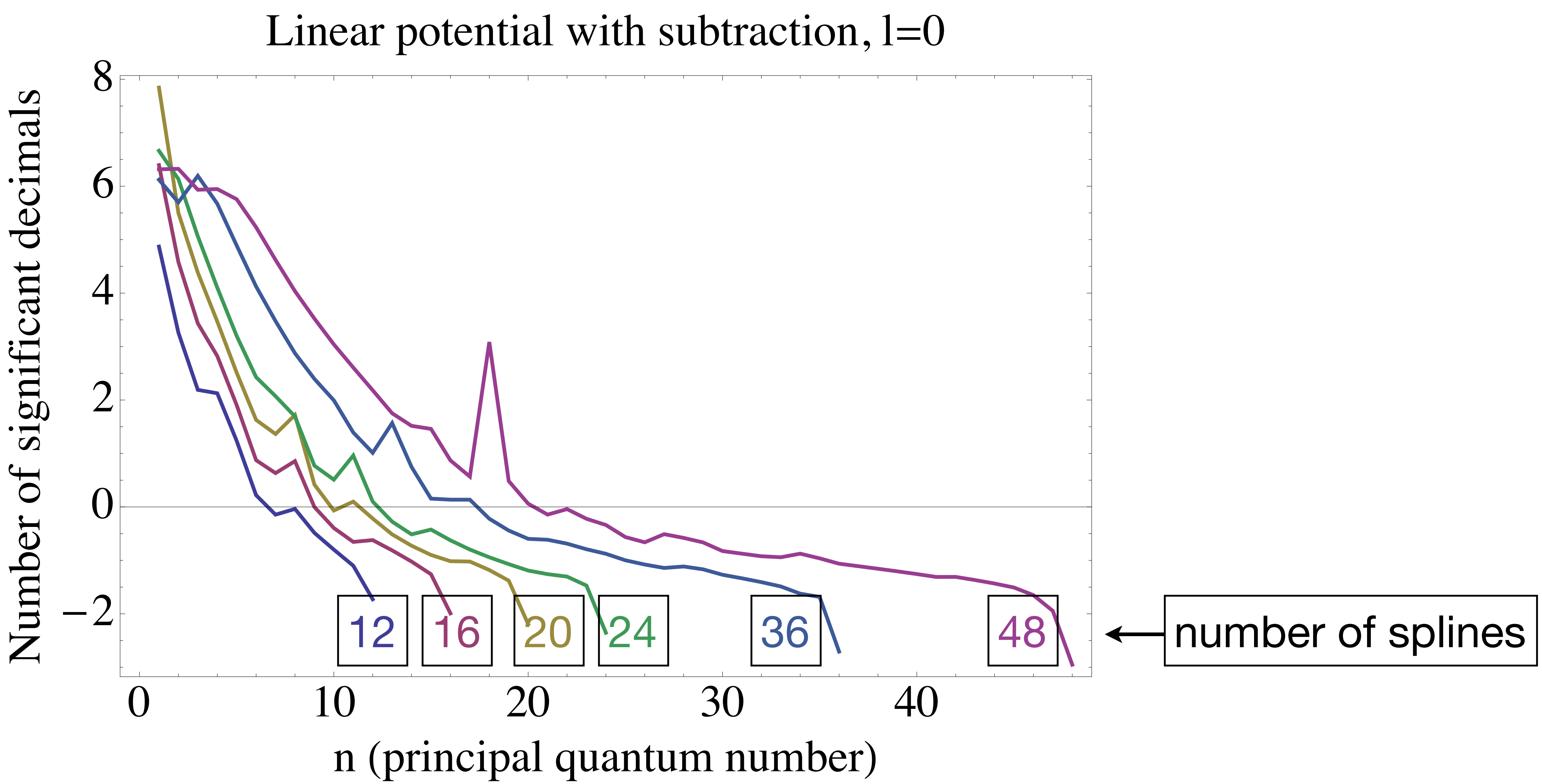
$$E_n = -z_n \left(\frac{\sigma^2}{2m_R} \right)^{1/3} \qquad \mathrm{Ai}(z_n) = 0$$

We find very good convergence of our numerical results to the exact energies

Obtained wave functions coincide with Fourier transformed exact r-space solutions (Airy funcions)

Accuracy of energy levels

Accuracy of energy levels decreases with n ⟶ increase number of splines if needed



Arbitrary partial waves

The partial wave projected Schrödinger equation with a linear potential has a simple form:

$$\frac{p^2}{2m_R}\psi_{lm}(p) + \mathrm{P}\!\int_0^\infty \frac{dk\,k^2}{(2\pi)^3}\left[\langle p\,lm|V_A|k\,lm\rangle\psi_{lm}(k) - \langle p\,00|V_A|k\,00\rangle\psi_{lm}(p)\right] = E\psi_{lm}(p)$$

The matrix element of V_A in partial wave $\{lm\}$ can be written as

$$\langle p\,lm|V_A|k\,lm\rangle = 2\pi(-8\pi\sigma)\left[\frac{2P_l(y)}{(p^2-k^2)^2} - \frac{P_l'(y)}{(2pk)^2}\ln\left(\frac{p+k}{p-k}\right)^2 + \frac{2w_{l-1}'(y)}{(2pk)^2}\right]$$

where

$$y = \frac{p^2+k^2}{2pk}$$

$$w_{l-1}(y) = \sum_{m=1}^{l}\frac{1}{m}P_{l-m}(y)P_{m-1}(y)$$

Most singular term as in S-wave

Can be subtracted in analogous way

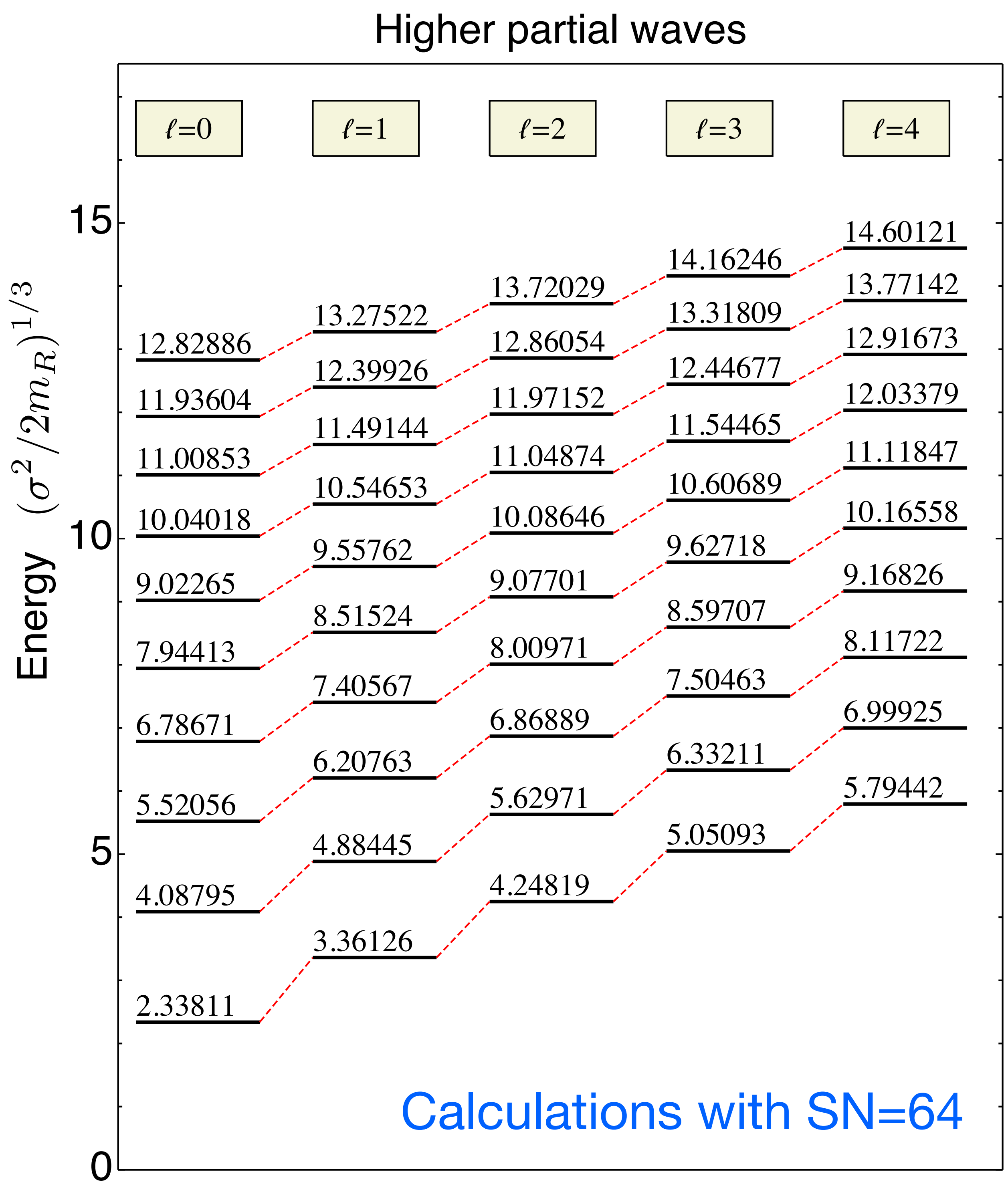
Additional logarithmic singularity for $l \geqslant 1$

Integrable, but numerically much easier after subtraction

Singularity-free partial-wave Schrödinger equation:

$$\left[\frac{p^2}{2m_R} + \frac{\sigma\pi l(l+1)}{4p}\right]\psi_l(p) - \frac{2\sigma}{\pi}\int_0^\infty dk\left\{\frac{1}{k^2-p^2}\left[\frac{2k^2}{k^2-p^2}\Big(P_l(y)\psi_l(k)-\psi_l(p)\Big) - p\psi_l'(p)\right] + \frac{w_{l-1}'(y)}{2p^2}\psi_l(k) - \frac{1}{4p^2}\ln\left(\frac{p+k}{p-k}\right)^2\left[P_l'(y)\psi_l(k) - \frac{l(l+1)}{2}\frac{p}{k}\psi_l(p)\right]\right\} = E\psi_l(p)$$

Higher partial waves



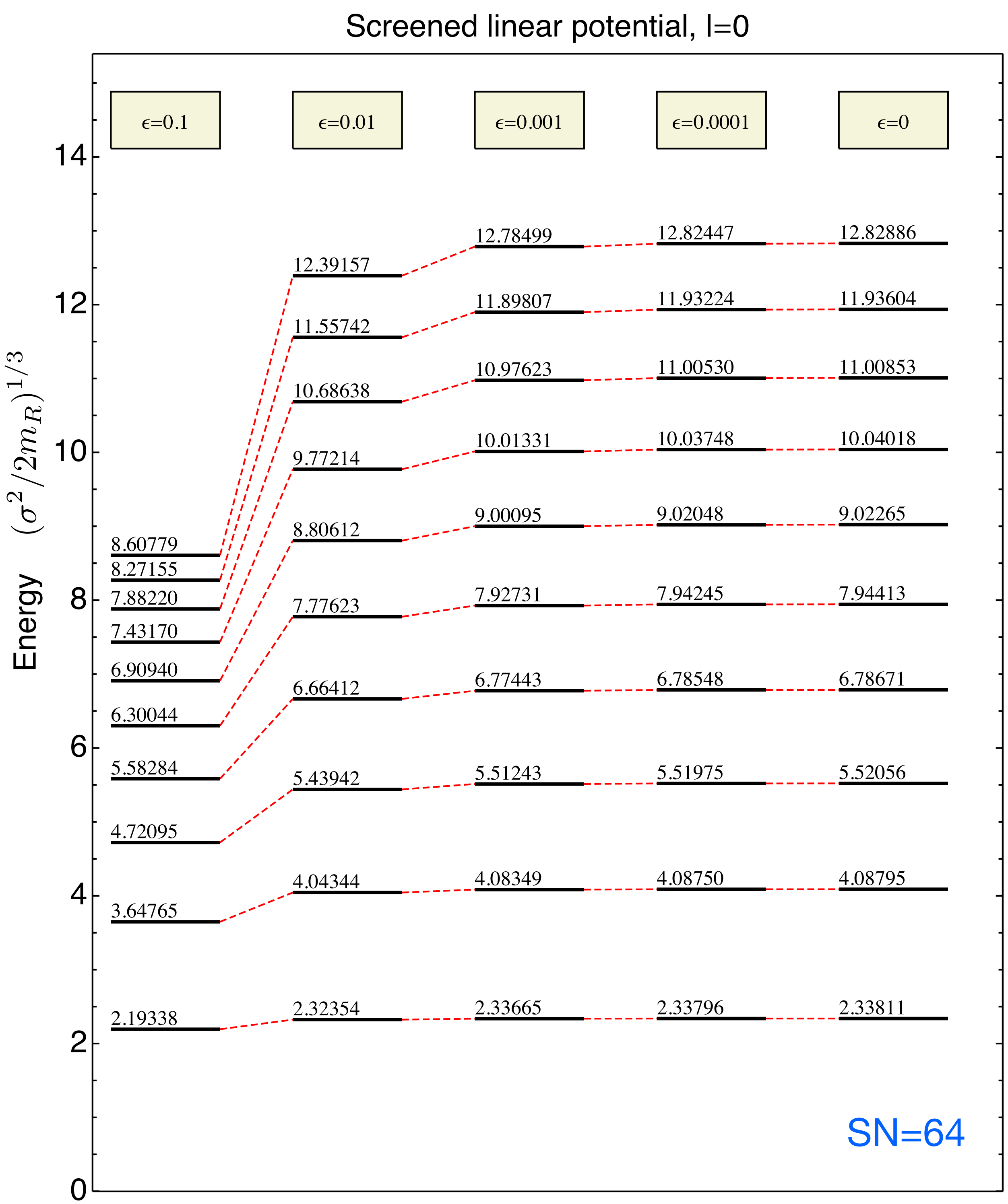
With singularity-free equation:

Numerical results for higher partial waves are also very accurate and stable

Of course, convergence becomes slower as ℓ or n increase

Screened linear potential in momentum space

Another test: take the unscreened limit numerically



The kernel is regular for finite ε

but

The problem is not trivial: for small ε the kernel is almost singular

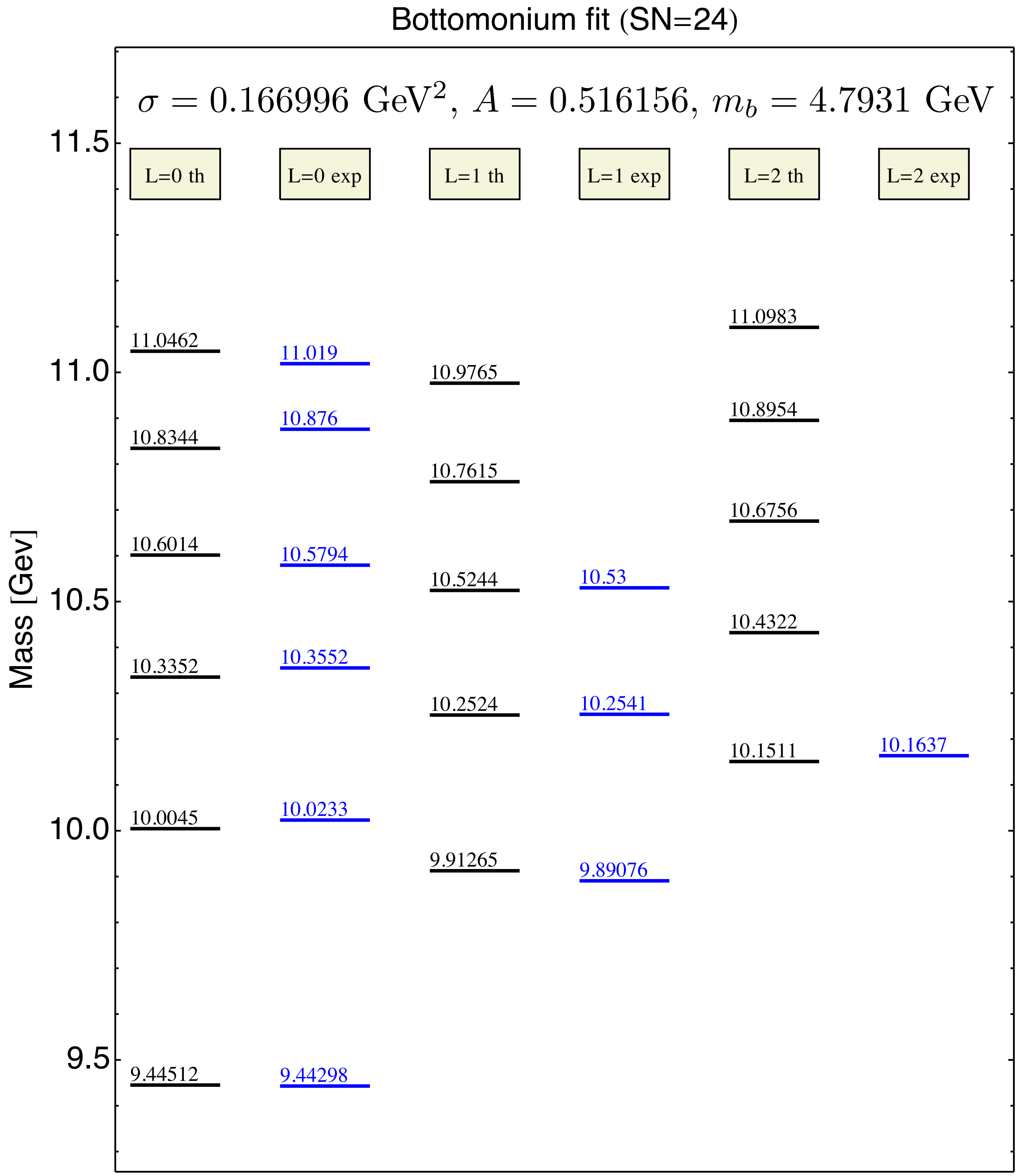
For typical numerical problems see, e.g., Chen, PRD **86**, 036013 (2012)

With our equations, the unscreened limit is reached smoothly

The screened linear potential might also be of interest by itself: simulate string breaking at large r

Bottomonium

Simple application: fit the bottomonium spectrum with a Cornell-type (funnel) potential



$$W(r) = \sigma r - \frac{A}{r}$$

but solved in momentum space!

No spin-dependence in model:
fit to spin-averaged experimental states

Three parameters: $\sigma,\ A,\ m_b$

We obtain a good fit
→ numerical method works well
for linear + Coulomb potential

Summary

- Presented covariant quark-antiquark bound-state equations for mesons in the Covariant Spectator Theory
- Momentum space formulation with Minkowski metric
- Singularities in kernel from linear confining interaction can be removed (subtraction techniques)
- Numerical methods tested in the nonrelativistic limit
- We obtain accurate and stable solutions
- Very good method for the nonrelativistic problem, but it should also work well in CST

For more details: see Elmar Biernat's talk!