FIXED POINTS OF THE SIMILARITY RENORMALIZATION GROUP AND THE NUCLEAR MANY BODY PROBLEM

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References

- Implicit vs Explicit Renormalization and Effective Interactions
  e-Print: arXiv:1307.1231
- Long distance symmetries for nuclear forces and the similarity renormalization group
  AIP Conf.Proc. 1520 (2013) 346-348
- Nuclear Symmetries of the similarity renormalization group for nuclear forces
  PoS CD12 (2013) 106
- Symmetries of the Similarity Renormalization Group for Nuclear Forces
  Phys.Rev. C86 (2012) 034002

Introduction

- How much do we need to know light nuclei to predict heavy nuclei ?
- Nucleon size $a \sim 1\text{fm}$
- Nuclear Force $\sim 1/m_\pi = 1.4\text{fm}$
- Nuclear matter (interparticle distance)

$$\rho_{nm} = 0.17\text{fm}^{-3} = \frac{1}{(1.8\text{fm})^3}$$

- Fermi Momentum

$$k_F = 270\text{MeV} \qquad \lambda_F = \pi/k_F = 2.3\text{fm} \gg 1/\sqrt{m_\pi M_N} = 0.5\text{fm}$$

1. Can we ignore explicit core and explicit (and/or chiral) pions ? → R. Navarro Pérez
2. What are the errors in the interaction →J. E. Amaro

- Nuclear many body Hamiltonian H

$$H = \sum_i T_i + \sum_{i<j} V_{2,ij} + \sum_{i<j<k} V_{3,ijk} + \sum_{i<j<k<l} V_{4,ijkl} + \dots$$

- NN: $V_{2,ij}$ (deuteron+NN scattering data)
- 3N: Triton+ N-deuteron scattering
- 4N: $\alpha-$particle, dd,tp etc, scattering
- Chiral hierarchy of few body multipionic forces (Weinberg)
- Typical Range of multinucleon forces $e^{-m_\pi d} \sim 0.2$

$$V_{NN} \sim e^{-m_\pi d} \quad V_{NNN} \sim e^{-2m_\pi d} \quad V_{NNNN} \sim e^{-3m_\pi d}$$

- Typical NN wavelengths $\ge 1/\sqrt{m_\pi M_N} \sim 0.5\mathrm{fm}$

$\rightarrow$ Few wavelengths within a range
(Coarse grained Effective interactions)

The off-shell problem

- Two-body NN Interactions are not uniquely determined by perfect scattering data, or spectrum.
- How large is the ambiguity ?
- Polyzou-Glöckle (Few Body System 1990)
  1. “Different off-shell extensions of two-body forces can be equivalently realized as three-body interactions”
  2. “There are no experiments measuring only three-body binding energies and phase shifts that can determine if there are no three-body forces in a three-body system.”
  3. “There may be some systems for which it is possible to find a representation in which three-body forces are not needed.”
- Linear correlation (Tjon line) between triton and α particle binding energy keeping two body scattering fixed

$$B_\alpha = aB_t + b$$

Isospectral flow in SRG

- Wilson-Glazek generator is unitary

$$\frac{dV_s}{ds} = [[T, H_s], H_s] = [[T, V_s], T + V_s] \to \mathrm{Tr} H_s^n = \mathrm{Tr} H_0^n$$

- Convergence in Frobenius norm and metric (potentials can be compared)

$$||V||^2 \equiv \mathrm{Tr} V^2 \qquad d(A, B) \equiv ||A - B||$$

- Monotonous decrease

$$\frac{d}{ds} \mathrm{Tr} V_s^2 = 2\mathrm{Tr}[T, V_s]^2 = -2\mathrm{Tr}[T, V_s]^\dagger [T, V_s] \le 0$$

$$s_0 < s \qquad 0 < \mathrm{Tr} V_s^2 \le \mathrm{Tr} V_{s_0}^2$$

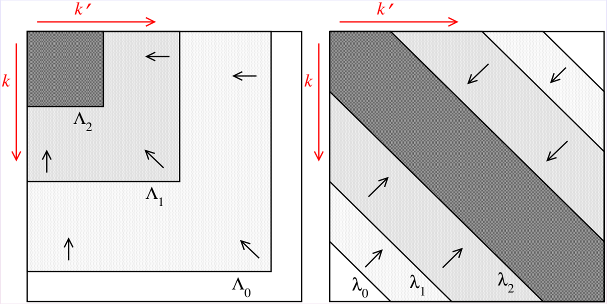
- Limiting Potential is the smallest possible with the same spectrum

$$\lim_{s\to\infty} \mathrm{Tr} V_s^2 = \min_V \mathrm{Tr} V^2 \Big|_{T+V=UH_0U^\dagger}$$

- High energy states are enhanced by Frobenius norm

$$1 = \frac{2}{\pi} \int_0^\infty p^2 dp |p\rangle\langle p| \to \mathrm{Tr} V^2 = \left(\frac{2}{\pi}\right)^2 \int_0^\infty p^2 dp \int_0^\infty k^2 dk |V(p, k)|^2$$

Integrating out vs Similarity Renormalization Group



- $V_{\rm lowk} \to$ Scattering reproduced until the cut-off.

$$\delta_{\rm lowk}(k,\Lambda) = \delta(k)\theta(\Lambda - k)$$

- $V_{\rm SRG}$ Scattering reproduced at ALL eneries.

$$\delta_{\rm SRG}(k,\lambda) = \delta(k)$$

Operator space

- In NN system most states are continuum states (except deuteron)
- Equations need discretization and cut-off in momentum space

$$p_n \quad (n = 1, \ldots, N) \rightarrow \Delta p_n \equiv w_n \rightarrow p_{\max} = \Lambda$$

- Closure relation

$$1 = \frac{2}{\pi} \sum_{n=1}^{N} w_n p_n^2 |p_n\rangle \langle p_n|$$

- Standard matrix multiplication

$$\bar{A}_{nm} = \frac{2}{\pi} p_n \sqrt{w_n} A_{nm} p_m \sqrt{w_m} \rightarrow \langle A, B \rangle = \sum_{n,m=1}^{N} \bar{A}^*_{nk} \bar{B}_{kn}$$

- SRG equations

$$\frac{d\bar{V}_{nm}}{ds} = -(e_n - e_m)^2 \bar{V}_{nm} + \sum_k (e_n + e_m - 2e_k) \bar{V}_{nk} \bar{V}_{km}$$

Fixed points and stability analisis

- Fixed points (Wilson)

$$\frac{d}{ds}\sum_{nm}|V_{nm}|^2 = -\sum_{nm}|V_{nm}|^2(\epsilon_n-\epsilon_m)^2 = 0 \to V_{nk} = V_n\delta_{nk}$$

- Energy eigenvalues

$$H\psi_n = E_n\psi_n \equiv (\epsilon_n + V_n)\psi_n$$

- Perturbation around the equilibrium point

$$V_{nk} = V_n\delta_{nk} + \Delta_{nk} \to \Delta V'_{nk} = -\Delta V_{nk}(\epsilon_n-\epsilon_m)(E_n-E_m)$$

- Only ordered as free ones are asymptotically stable (crossing forbbiden)

$$H_{nm}(s) = E_n\delta_{n,m} + C_{nm}e^{-(\epsilon_n-\epsilon_m)(E_n-E_m)s} + \dots$$

- LS equation on the grid

$$R_{ij} = V_{ij} + \sum_{k \neq i} \frac{2}{\pi} w_k \, p_k^2 \frac{R_{ik} V_{kl}}{p_i^2 - p_k^2}$$

- Phase shifts

$$R_{nn} = -\frac{\tan \delta_n^{\rm LS}}{p_n} \equiv V_n$$

- Limiting potential has no off-shellness

$$\lim_{\lambda \to 0} V_{nm}(\lambda) = -\frac{\tan \delta_n^{\rm LS}}{p_n} \delta_{nm}$$

- However, the LS phase shifts are not independent of λ in a finite grid

$$\delta(p_n, \lambda) \neq \delta(p_n, \lambda')$$

Wegner generator

- Evolution equation

$$\frac{dH}{ds} = [[H_D, H], H] \qquad H_D = \mathrm{diag} H$$

- $$\frac{d}{ds}\mathrm{Tr}(H - H_D)^2 = 2\mathrm{Tr}[H_D, H]^2 = -2\mathrm{Tr}[H_D, H]^\dagger [H_D, H] \leq 0$$

  so that $||H - H_D|| \to 0$

$$\lim_{s\to\infty} H = H_D = \min_{H=UH_0U^\dagger} ||H - H_D||$$

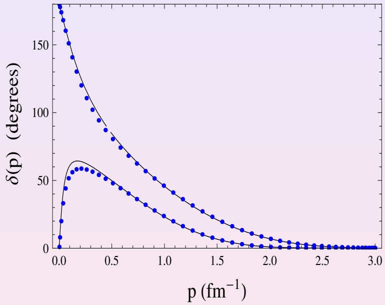
- Wilson generator and Wegner generators provide the same final fixed points up tp permutations
- Wegner generator (all points are stable, crossing allowed)

$$H_{nm}(s) = E_n \delta_{n,m} + C_{nm} e^{-(E_n - E_m)^2 s} + \dots$$

Toy model for S-waves

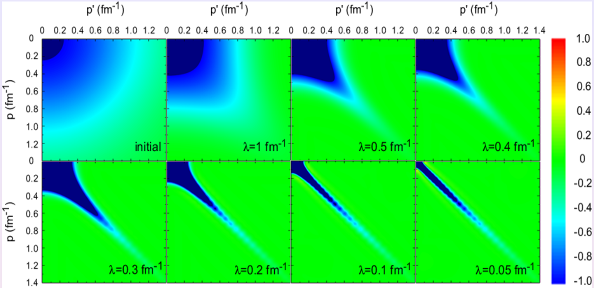
- Separable interaction

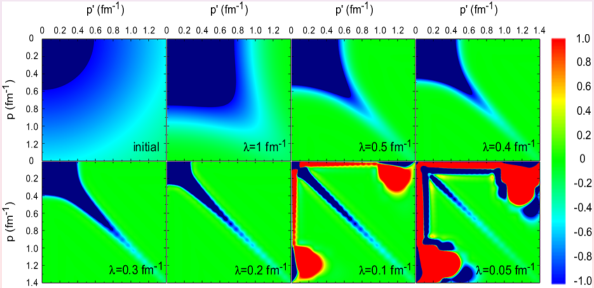
$$V_\alpha(p,p') = C_\alpha e^{-(p^2+p'^2)/L_\alpha^2} \qquad \alpha = {}^1S_0, {}^3S_1 \tag{1}$$



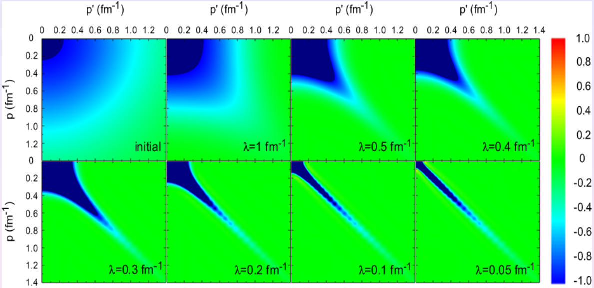
Parameter Units	α_0 (fm)	r_0 (fm)	C (fm)	L (fm^{-1})
$^1S_0$	-23.74	2.77	-1.9158	0.6913
$^3S_1$	5.42	1.75	-2.3006	0.4151

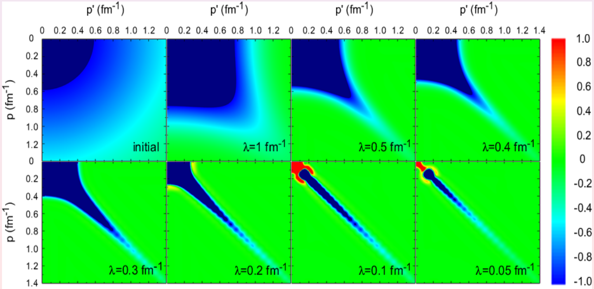
SRG evolution (Wilson generator)



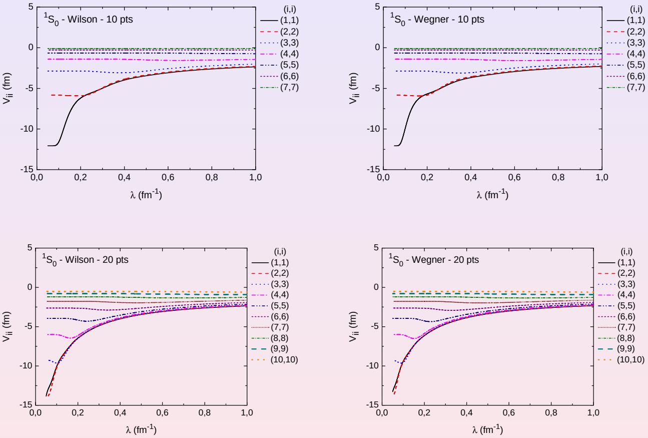


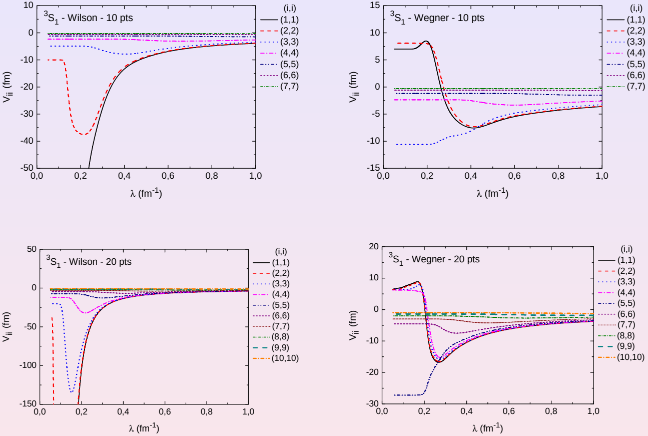
SRG evolution (Wegner generator)





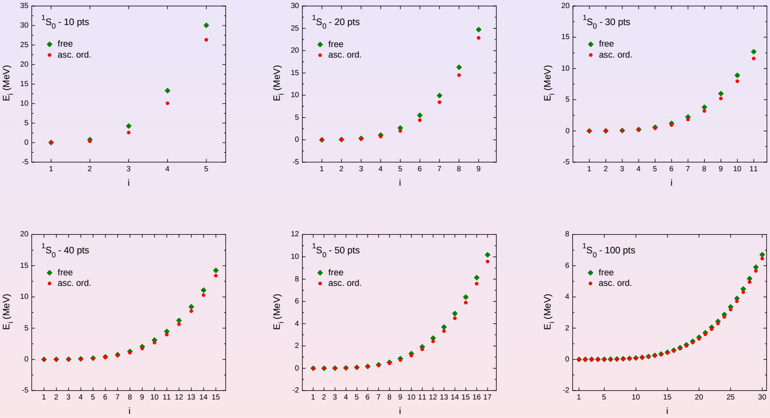
Diagonal Matrix Elements Evolution



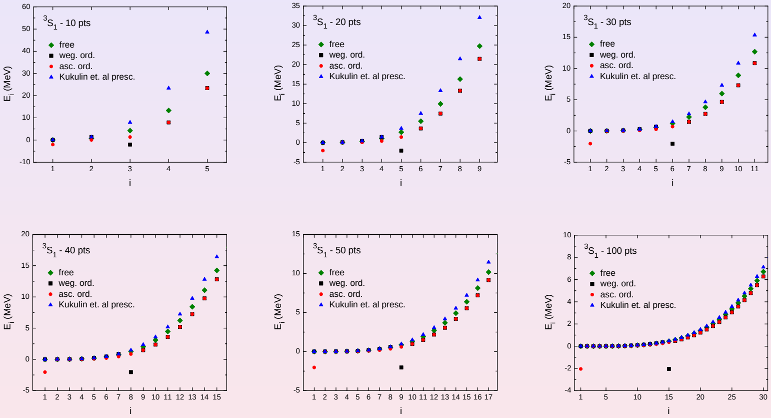


$^3S_1$ - Wilson - 10 pts
$^3S_1$ - Wegner - 10 pts
$^3S_1$ - Wilson - 20 pts
$^3S_1$ - Wegner - 20 pts
V_{ii} (fm)
λ (fm^{-1})
(i,i)
(1,1)
(2,2)
(3,3)
(4,4)
(5,5)
(6,6)
(7,7)
(8,8)
(9,9)
(10,10)

Eigenvalues ordering



Eigenvalues ordering



Binding Energies

- Mean field Slater Determinant

$$\psi(\vec{p}_1, \ldots, \vec{p}_A) = \mathcal{A}\left[\phi_{n_1,l_1,s,m_{s1},t,m_{t_1}}(\vec{p}_1)\ldots\phi_{n_A,l_A,s,m_{sA},t,m_{t_A}}(\vec{p}_A)\right] . \quad (2)$$

- Single particle states (Harmonic oscillator)

$$P_{nl}(p) = N_{nl} e^{-\frac{1}{2}b^2p^2} (bp)^l L_{n-1}^{l+\frac{1}{2}}\left(b^2p^2\right) \quad (3)$$

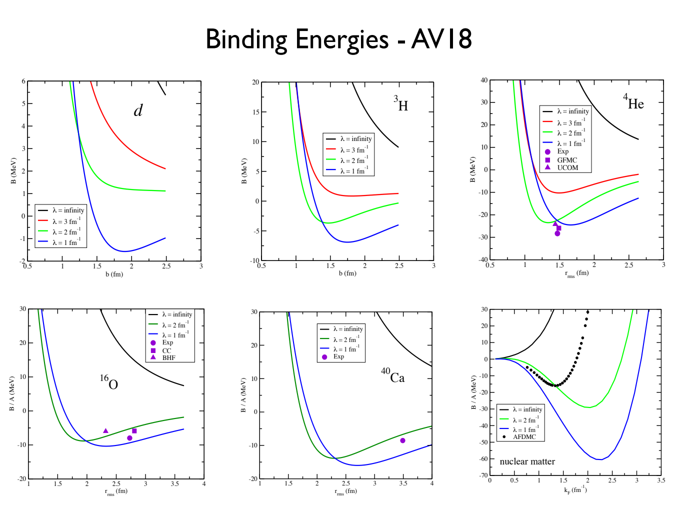
- Two body interaction (Talmi-Moshinsky)

$$\langle V_2 \rangle_A = \sum_{nlJS} g_{nlJS} \langle nl | V^{JST} | nl \rangle , \quad (4)$$

- Nuclei: Shell model (mean field)

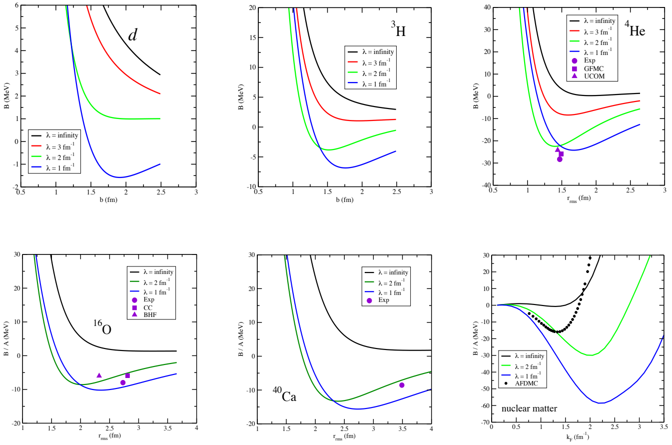
$$\mathrm{d} : (1s)^2 \quad \mathrm{t} : (1s)^3 \quad {}^4\mathrm{He} : (1s)^4 ,$$
$$ {}^{16}\mathrm{O} : (1s)^4(1p)^{12} \quad {}^{40}\mathrm{Ca} : (1s)^4(1p)^{12}(2s)^4(1d)^{20}$$

Binding Energies



Binding Energies

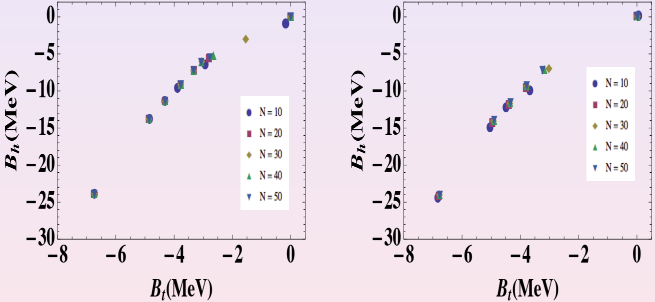
Binding Energies - N3LO



SRG Correlations

- The Wilson and Wegner binding energy results for SRG evolved forces

$$\{-B_t, -B_\alpha\} = \min_b \left[(A-1)\langle \frac{p^2}{2M} \rangle + \frac{A(A-1)}{2} \frac{1}{2} \langle V_{1S0,\lambda} + V_{3S1,\lambda} \rangle \right] \Big|_{A=3,4}$$

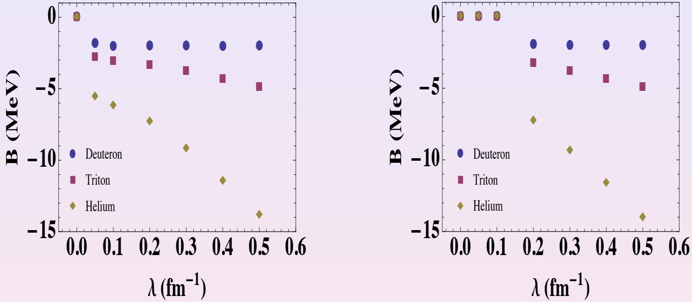


- Linear correlations in two regimes

$$\Delta B_\alpha / \Delta B_t \sim 2(\lambda \to 0) \qquad \Delta B_\alpha / \Delta B_t \sim 4(\lambda \sim 1)$$

The on-shell limit

- Wilson and Wegner generator results (N=50)



- On-shell results

$$\lim_{\lambda\to 0} E_t(\lambda) = -\frac{3}{2}B_d \qquad \lim_{\lambda\to 0} E_\alpha(\lambda) = -3B_d$$

SRG view of off-shellness and three-body force

- Isospectral transformations

$$\frac{dV_{ij}}{ds} = \left[\left[T_{ij}, V_{ij}\right], T_{ij} + V_{ij}\right], \quad (5)$$

$$\begin{aligned}\frac{dV_{123}}{ds} &= [[T_{12}, V_{12}], V_{13} + V_{23} + V_{123}] \\ &+ [[T_{13}, V_{13}], V_{12} + V_{23} + V_{123}] \\ &+ [[T_{23}, V_{23}], V_{12} + V_{13} + V_{123}] \\ &+ [[T_{\rm rel}, V_{123}], H_s]. \quad (6)\end{aligned}$$

- What is the initial condition ?
- Final condition is unique

$$[T_{12}, V_{12}] = 0 \qquad [T_{\rm rel}, V_{123}] = 0 \quad (7)$$

- Diagonal potential in momentum space (no off-shellness)

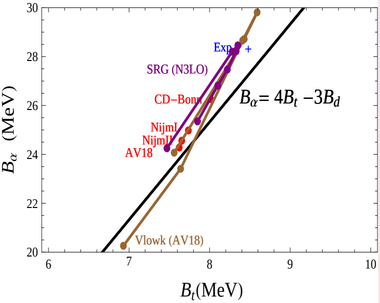
Correlations with on-shell 3-body forces

- The on-shell triton (3 doublets) and α (6 doublets) binding

$$-B_t = -\underbrace{\frac{3}{2}B_d}_{3.3\mathrm{MeV}} + \underbrace{\langle t|V_3|t\rangle}_{\mathrm{off-shellness}} \qquad -B_\alpha = -\underbrace{3B_d}_{6.6\mathrm{MeV}} + \underbrace{\langle \alpha|V_3|\alpha\rangle}_{\mathrm{off-shellness}}$$

- Taking $\langle \alpha|V_3|\alpha\rangle = 4\langle t|V_3|t\rangle$ (4 triplets)

$$\begin{aligned} B_\alpha &= 4B_t - 3B_d \\ &= 4 \times 8.482 - 3 \times 2.225 = 27.53\,(\mathrm{exp.}28.296)\,\mathrm{MeV} \end{aligned}$$



Conclusions

1. SRG methods allow to reduce off-shell ambiguity completely
2. Only measurable two-body information is needed
3. Simple explanation of the observed linear correlations (Tjon line)
4. On-shell 3-body forces are large and 4-body forces are moderate
5. Extension to other nuclei, neutron and nuclear matter is possible