Ultracold Collisions and Universal Correlations*

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***based on joint work with* A. K. Motovilov and W. Sandhas**

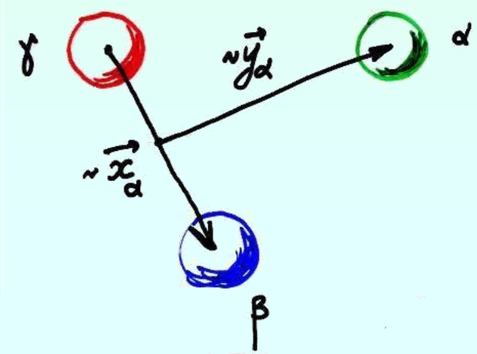
Outline

- Formalism (Faddeev equations)
- Phillips line and modified Phillips line
- Helium trimer and nnp system

In describing the three-body system we use the standard Jacobi coordinates [4] $\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha$, $\alpha = 1, 2, 3$, expressed in terms of the position vectors of the particles $\boldsymbol{r}_i \in \mathbb{R}^3$ and their masses m_i,

$$\boldsymbol{x}_\alpha = \left[\frac{2m_\beta m_\gamma}{m_\beta + m_\gamma}\right]^{1/2} (\boldsymbol{r}_\beta - \boldsymbol{r}_\gamma)$$

$$\boldsymbol{y}_\alpha = \left[\frac{2m_\alpha(m_\beta + m_\gamma)}{m_\alpha + m_\beta + m_\gamma}\right]^{1/2} \left(\boldsymbol{r}_\alpha - \frac{m_\beta \boldsymbol{r}_\beta + m_\gamma \boldsymbol{r}_\gamma}{m_\beta + m_\gamma}\right)$$



where (α, β, γ) stands for a cyclic permutation of the indices $(1, 2, 3)$. The coordinates $\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha$ fix the six-dimensional vector $X \equiv (\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha) \in \mathbb{R}^6$. The vectors $\boldsymbol{x}_\beta, \boldsymbol{y}_\beta$ corresponding to the same point X as the pair $\boldsymbol{x}_\alpha, \boldsymbol{y}_\alpha$ are obtained using the transformations

$$\boldsymbol{x}_\beta = c_{\beta\alpha}\boldsymbol{x}_\alpha + s_{\beta\alpha}\boldsymbol{y}_\alpha \qquad \boldsymbol{y}_\beta = -s_{\beta\alpha}\boldsymbol{x}_\alpha + c_{\beta\alpha}\boldsymbol{y}_\alpha$$

where the coefficients $c_{\beta\alpha}$ and $s_{\beta\alpha}$ fulfil the conditions $-1 < c_{\beta\alpha} < +1$ and $s_{\beta\alpha}^2 = 1 - c_{\beta\alpha}^2$ with $c_{\alpha\beta} = c_{\beta\alpha}$, $s_{\alpha\beta} = -s_{\beta\alpha}$, $\beta \neq \alpha$ and depend only on the particle masses [4]. For equal masses $c_{\beta\alpha} = -\frac{1}{2}$.

[4] - L.D.Faddeev,S.P.Merkuriev, 1993, *Quantum scattering theory for several particles*

Three-body, theory formalism

$^4He_2$ - ^4He

Faddeev integro-differential equations after angular partial-wave analysis

$$\left[-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+l(l+1)\left(\frac{1}{x^2}+\frac{1}{y^2}\right)-E\right]F_l(x,y)$$

$$=\begin{cases}-V(x)\,\Psi_l(x,y), & x>c\\ 0, & x<c\end{cases}$$

At *L=0* the partial angular momentum l corresponds both to the dimer and an additional atom. x, y stand to the standard Jacobi variables.

$$\Psi_l(x,y)=F_l(x,y)+\sum_{l'}\int_{-1}^{1}d\eta\, h_{ll'}(x,y,\eta)\,F_{l'}(x',y'),$$

$$x'=(1/4x^2+3/4y^2-\sqrt{3}/2xy\eta)^{1/2},\; y'=(3/4x^2+1/4y^2+\sqrt{3}/2xy\eta)^{1/2},\; \eta=\hat{x}\cdot\hat{y}$$

The kernel $h_{ll'}$ depend only on hyperangles - see L.D.Faddeev,S.P.Merkuriev, 1993.

nnp system

S =3/2

$$(H_L - E)\Phi_L^q(x,y) = V_t(x)\Psi_L^q(x,y)$$

$$\Psi_L^q(x,y) = \Phi_L^q - \frac{1}{2}\int_{-1}^{1} du h^L(x,y,u)\Phi_L^q(x^{'},y^{'})$$

S=1/2

$$(H_L - E)\Phi_{1(2),L}^d(x,y) = V_{t(s)}(x)\Psi_{1(2),L}^d(x,y)$$

$$\Psi_L^d = \Phi_L^d + \frac{1}{2}\int_{-1}^{1} du h^L(u) B\Phi_L^d$$

$$\Phi_{i,L}^{q,d}(x,y)|_{x=0} = 0 \qquad \Phi_{i,L}^{q,d}(x,y)|_{y=0} = 0$$

L.D.Faddeev, S.P. Merkuriev, 1993

$^4He_2$ - ^4He

Three-body, theory formalism

Boundary conditions

$$F_l(x,y)_{x=0} = F_l(x,y)_{y=0} = 0,$$

Hard-core boundary conditions:

$$\Psi_l(x,y)_{x=c} = F_l(c,y) + \sum_{l'} \int_{-1}^{1} d\eta\, h_{ll'}(c,y,\eta)\ F_{l'}(x',y') = 0,$$

The asymptotic condition for the helium trimer bound states (as $\rho \to \infty$ and/or $y \to \infty$)

$$F_l(x,y) = \delta_{l0}\psi_d(x)\exp(i\sqrt{E_t - \varepsilon_d}\,y)[a_0 + o(y^{-1/2})] + \frac{\exp(i\sqrt{E_t}\rho)}{\sqrt{\rho}}[A_l(\theta) + o(\rho^{-1/2})]$$

Here, ψ_d is the dimer wave function, ε_d stands for the dimer energy,

$$\rho = \sqrt{x^2 + y^2}, \quad \theta = \arctan(x/y)$$

$^4He_2$ - ^4He

Three-body, theory formalism

Boundary conditions

The asymptotic condition for the partial-wave Faddeev components of the (2 + 1 → 2 + 1 ; 1 + 1 + 1) scattering wave function reads, (as $\rho \to \infty$ and/or $y \to \infty$)

$$\Phi_l(x,y;p) = \delta_{l0}\psi_d(x)\{\sin(py) + \exp(ipy)[a_0(p) + o(y^{-1/2})]\}$$

$$+ \frac{\exp(i\sqrt{E}\rho)}{\sqrt{\rho}}[A_l(\theta) + o(\rho^{-1/2})].$$

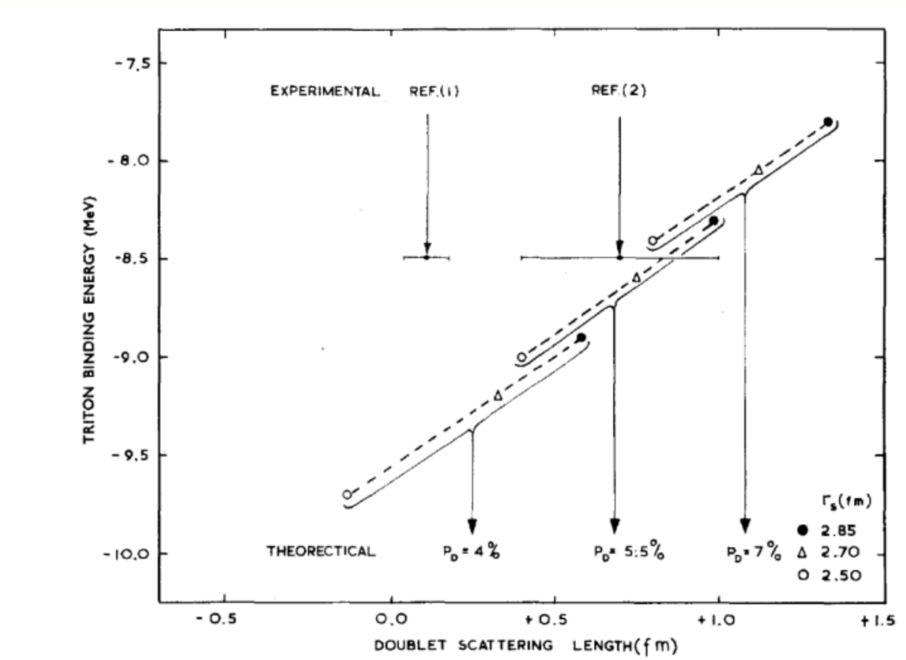
Here ψ_d is the dimer wave function, E stands for the scattering energy given by $E = \varepsilon_d + p^2$ with ε_d the dimer energy, and p is the relative momentum conjugate to the variable y. The coefficient $a_0(p)$ is nothing but the elastic scattering amplitude, while the functions $A_l(\theta)$ provides us, at $E > 0$, with the corresponding partial-wave Faddeev breakup amplitudes. The scattering length is given by

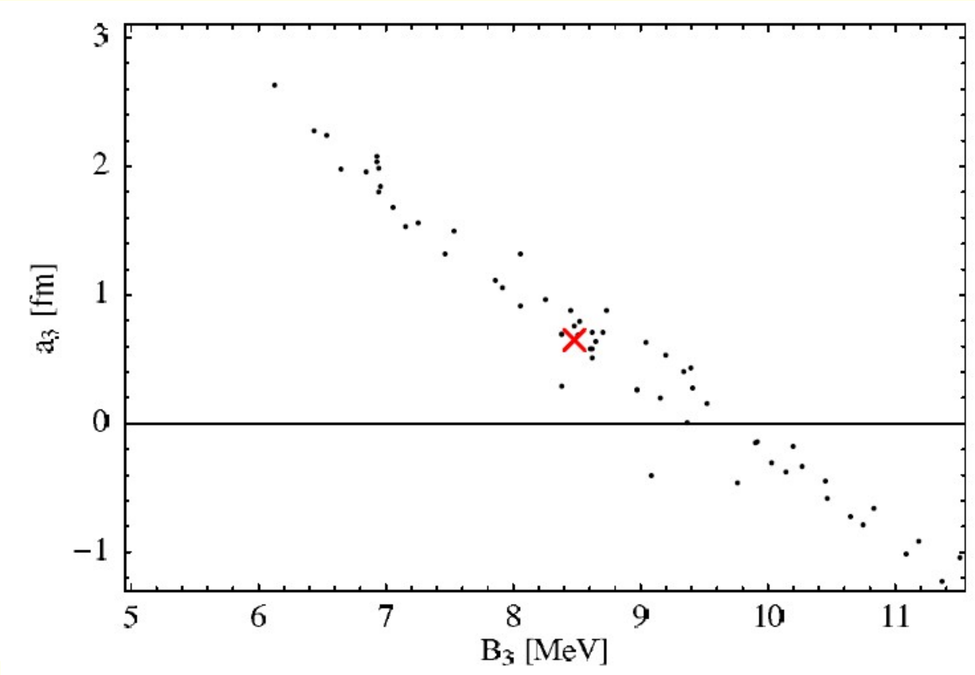
$$l_{sc} = -\frac{\sqrt{3}}{2}\lim_{p\to 0}\frac{a_0(p)}{p}$$

E.K, A.Motovilov,S.Sofianos
J.Phys.B **31**, 1279 (1998)

^{3}H - system

Three-body, theory



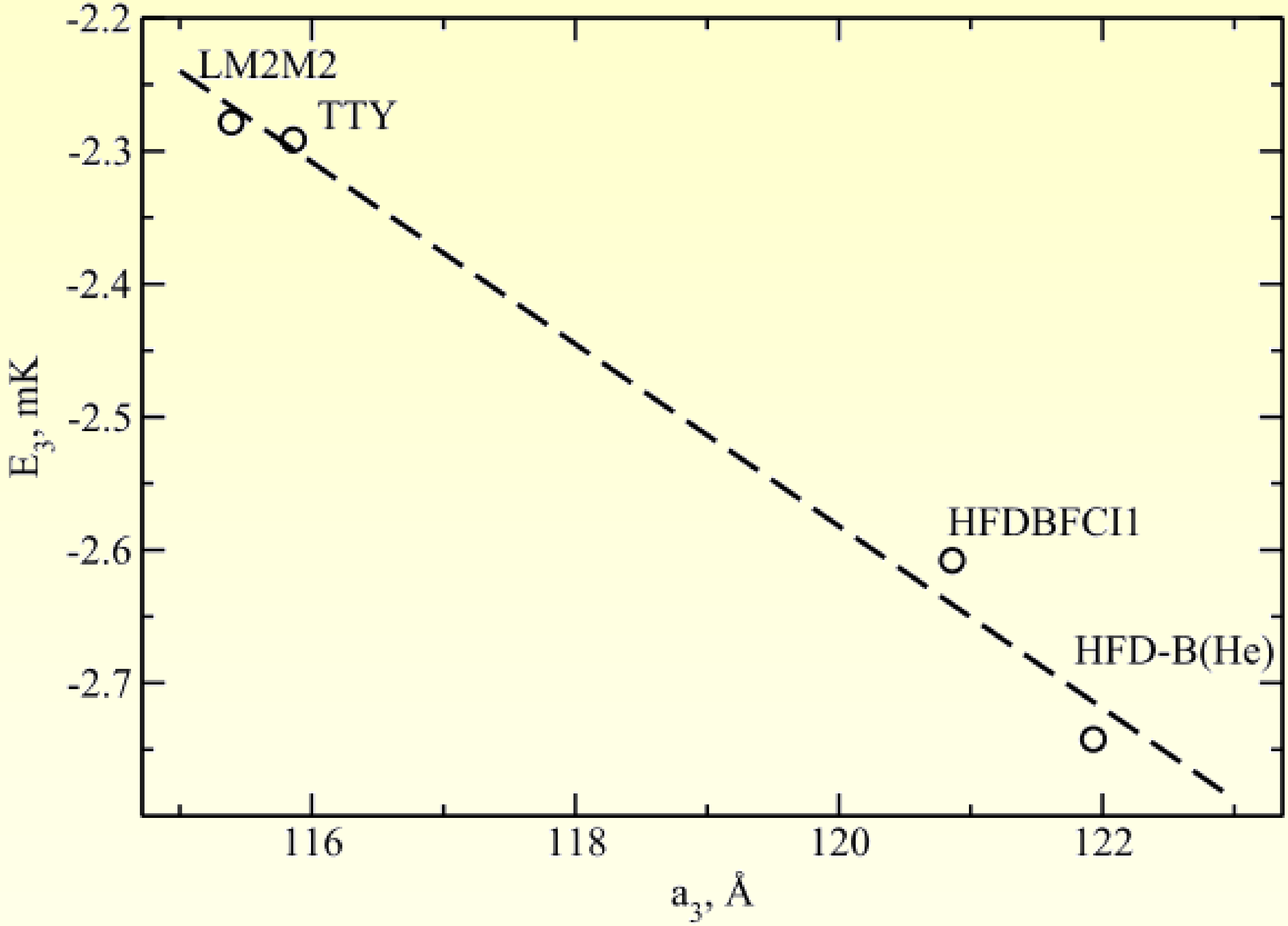


Phillips line from the original paper,
showing the unexpected linear correlation

A.C. Phillips Nucl. Phys A **107**, 209 (1968)

$^4He_3$

Three-body, theory

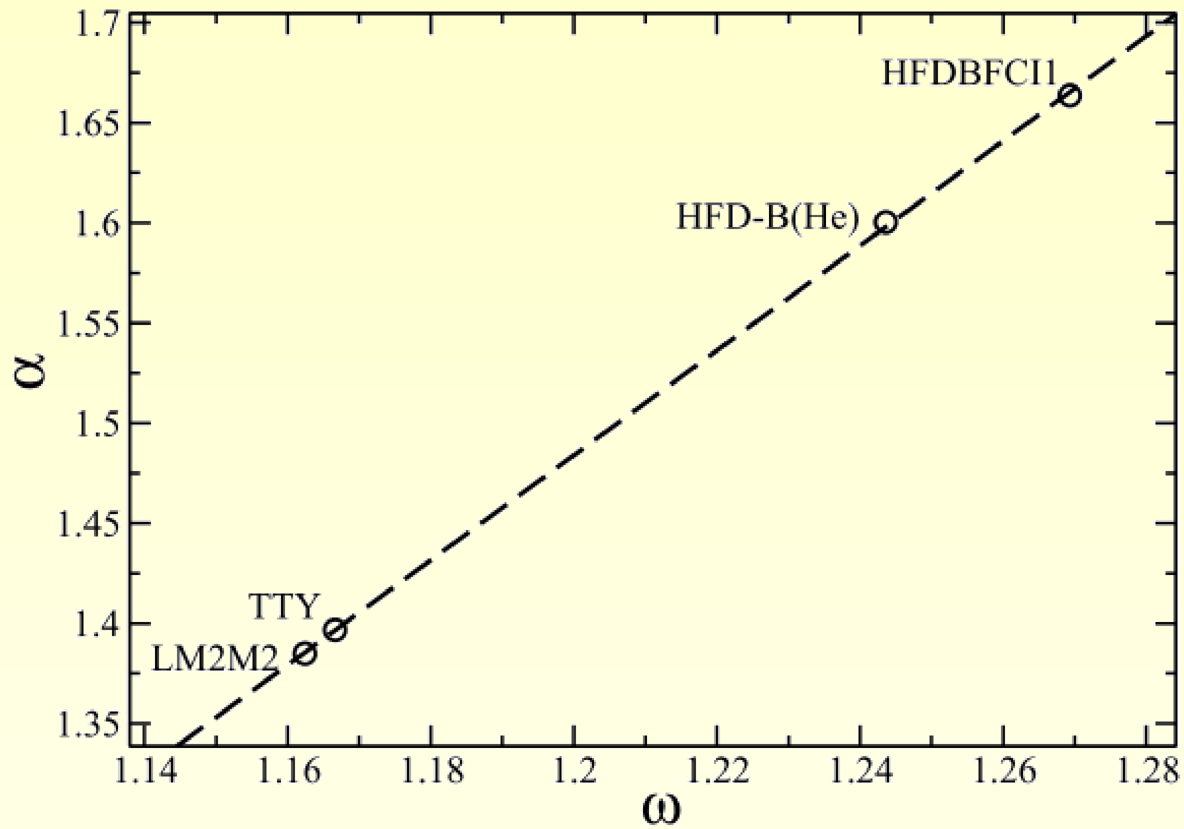


$^4He_2$ - ^4He

Three-body, theory

$$E_2 - E_3 \approx 1/(2m_{12}a_3^2)$$

V.Efimov, E.G.Tkachenko, *Phys.Lett. B* **157**, 108 (1985)



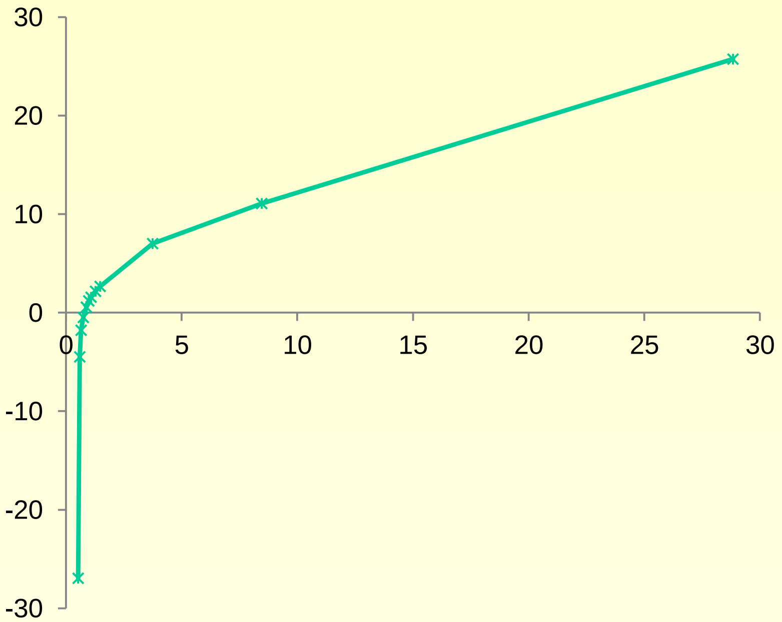
$$\alpha \equiv a_3\sqrt{-\mu E_2} \propto 1/\sqrt{E_3/E_2 - 1} \equiv \omega$$

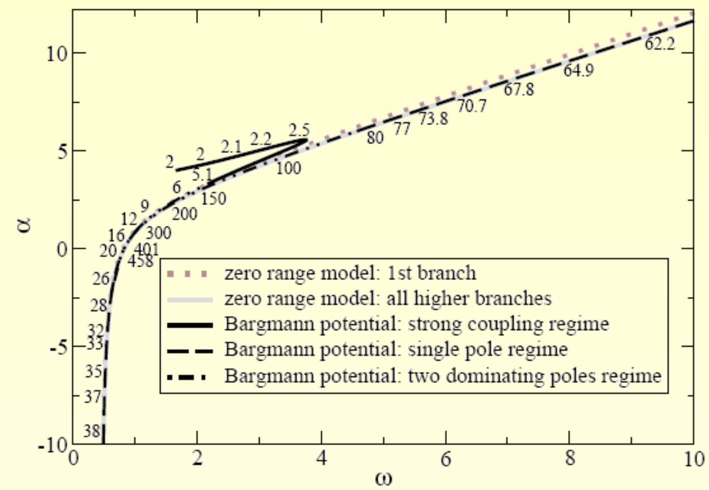
V.Roudnev, M.Cavagnero *Phys.Rev.Lett.* **108**, 110402 (2012)

$^4He_2$ - ^4He

Three-body, theory

$$V(x) = \lambda V_{HFD-B}(x)$$





$$\alpha \equiv a_3\sqrt{-\mu E_2} \propto 1/\sqrt{E_3/E_2 - 1} \equiv \omega$$

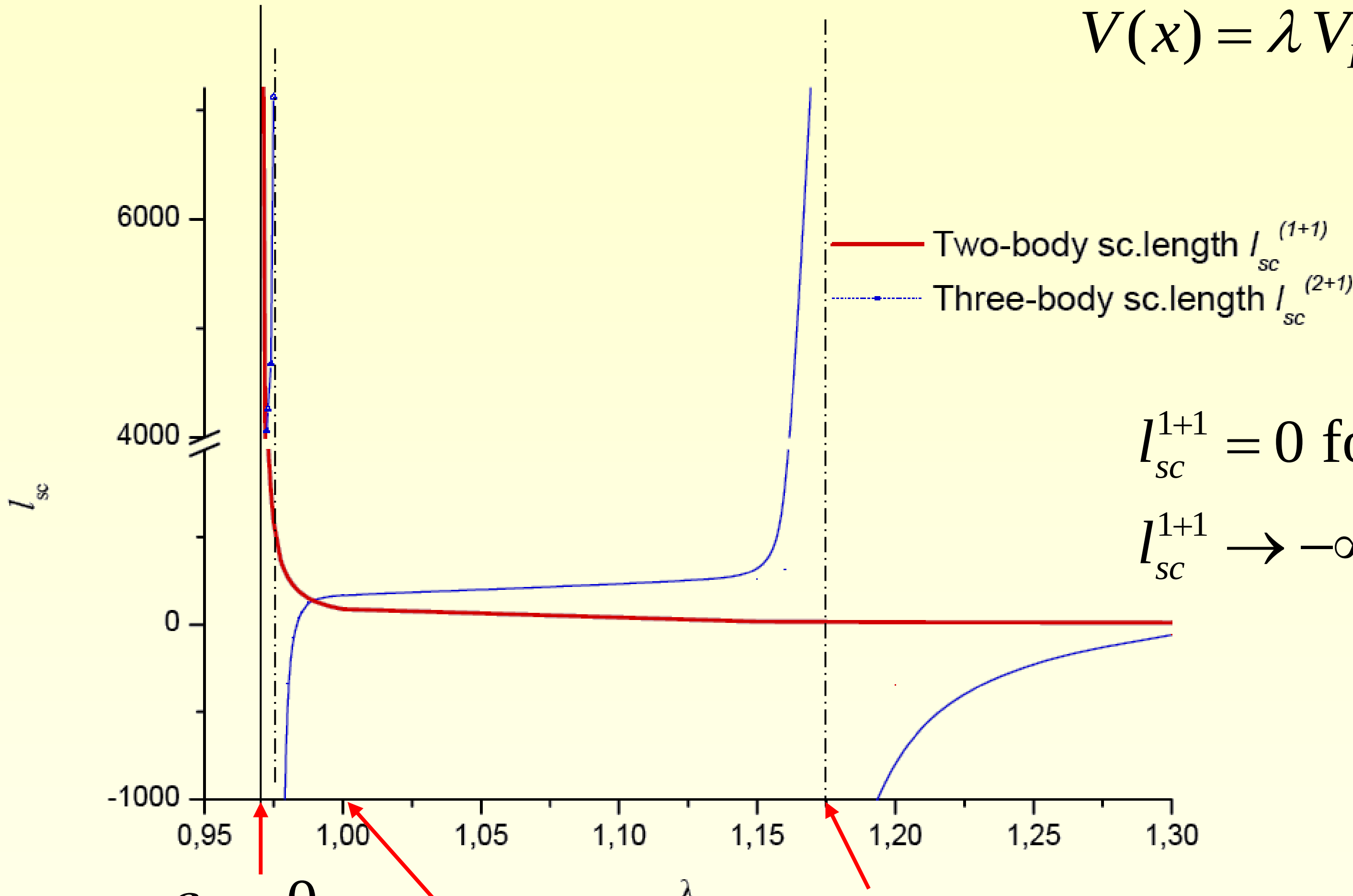
V.Roudnev, M.Cavagnero
Phys.Rev.Lett. **108**, 110402 (2012)

$^4He_2$ - ^4He

Three-body, theory scattering

Efimov-type properties of exited state

$$V(x) = \lambda V_{HFD-B}(x)$$



$$l_{sc}^{1+1} = 0 \text{ for } \lambda \approx 4.8$$

$$l_{sc}^{1+1} \to -\infty \text{ for } \lambda \approx 6.8$$

$\varepsilon_d = 0$

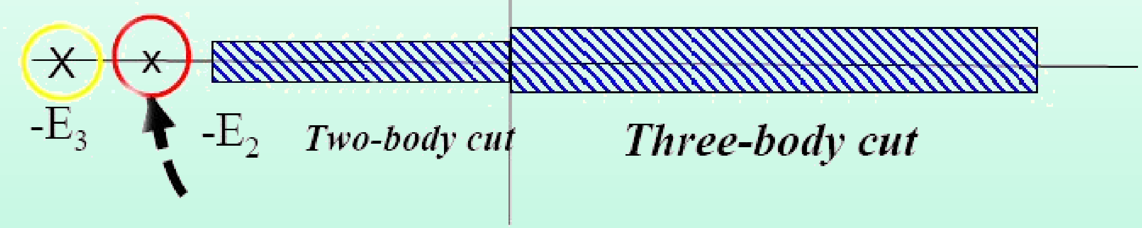
$^4He_3$

$E_{ex}^{(1)} = 0$

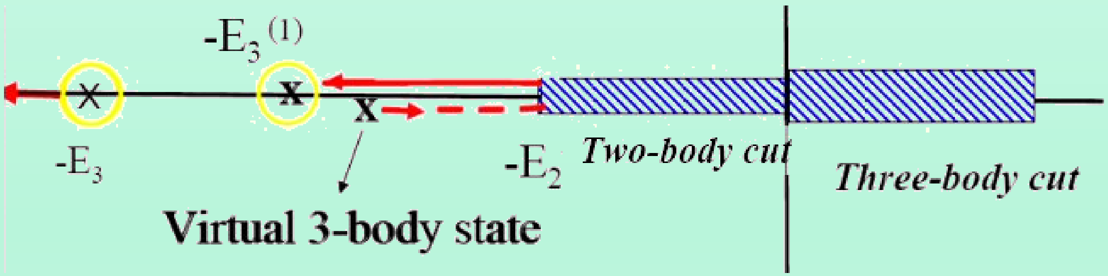
E.K, A.Motovilov, W.Sandhas
Nucl.Phys. A **790**, 752 (2007)

$^4He_3$

Three-body, theory resonances



$$V(x) = \lambda\, V_{HFD-B}(x)$$



λ	ϵ_d (mK)	$\epsilon_d - E_t^{(1)}$ (mK)	$\rho_{\max}$ (Å)
1.05	-12.244	0.873	300
1.10	-32.222	0.450	200
1.15	-61.280	0.078	150
1.16	-68.150	0.028	120
1.17	-75.367	0.006	120

λ	ϵ_d (mK)	$\epsilon_d - E_t^{(1)*}$ (mK)	$\rho_{\max}$ (Å)
1.18	-82.927	0.001	110
1.19	-90.829	0.016	110
1.20	-99.068	0.057	100
1.25	-145.240	0.588	85
1.30	-199.457	1.831	70
1.35	-261.393	3.602	70
1.40	-330.737	6.104	55
1.50	-490.479	12.276	50

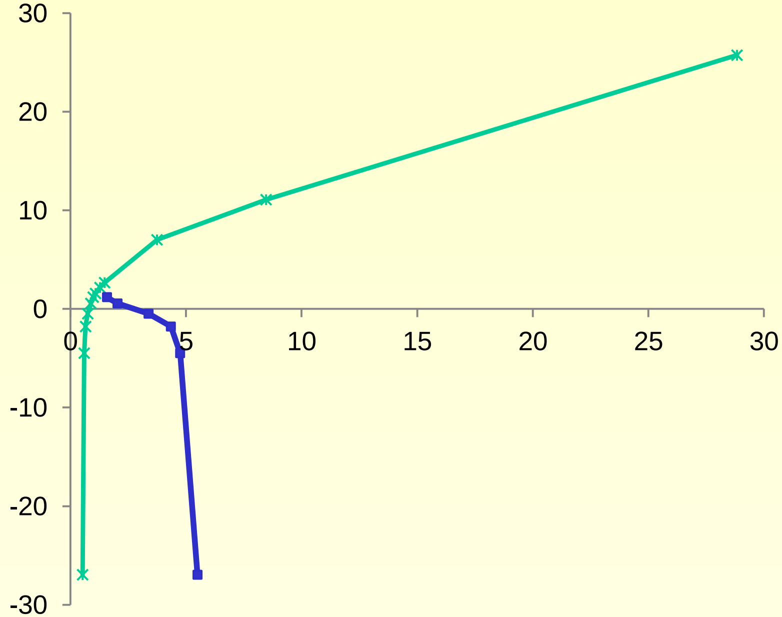
λ	ϵ_d	$\epsilon_d - E_{\rm ex}^{(1)}$	$\epsilon_d - E_{\rm virt}$	$\epsilon_d - E_{\rm ex}^{(2)}$	$l_{\rm sc}^{(1+2)}$	$l_{\rm sc}^{(1+1)}$	$\rho_{\max}$ (Å)
1.0	-1.685	0.773	–	–	160	88.6	700
0.995	-1.160	0.710	–	–	151	106	900
0.990	-0.732	0.622	–	–	143	132	1050
0.9875	-0.555	0.222	–	–	125	151	1200
0.985	-0.402	0.518	0.097	–	69	177	1300
0.982	-0.251	0.447	0.022	–	-75	223	1700
0.980	-0.170	0.396	0.009	–	-337	271	2000
0.9775	-0.091	0.328	0.003	–	-6972	370	3000
0.975	-0.036	0.259	–	0.002	7120	583	4500
0.973	-0.010	0.204	–	0.006	4260	1092	10000

E.A.K, Motovilov A.K.
Phys.At.Nucl. **60**, 235 (1997)

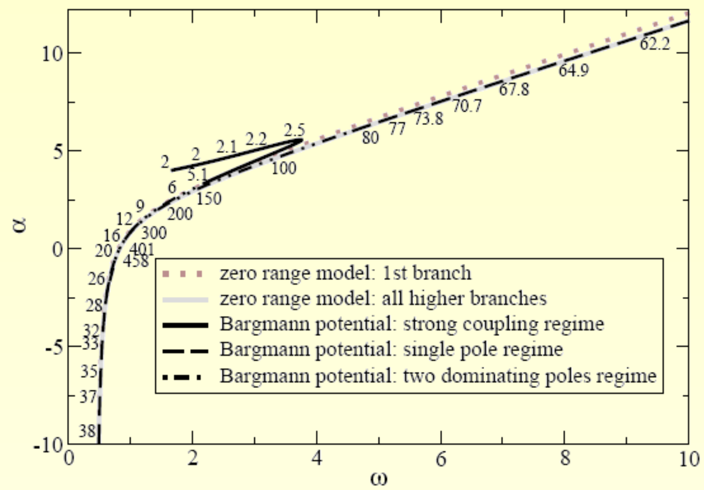
$^4He_2$ - ^4He

Three-body, theory

$$V(x) = \lambda V_{HFD-B}(x)$$



$$\alpha \equiv a_3\sqrt{-\mu E_2} \propto 1/\sqrt{E_3/E_2 - 1} \equiv \omega$$



Roudnev, Cavagnero
Phys.Rev.Lett. **108**, 110402 (2012)

$^4He_2$ - ^4He

Three-body, theory

S-matrix

$$\lambda = 1$$

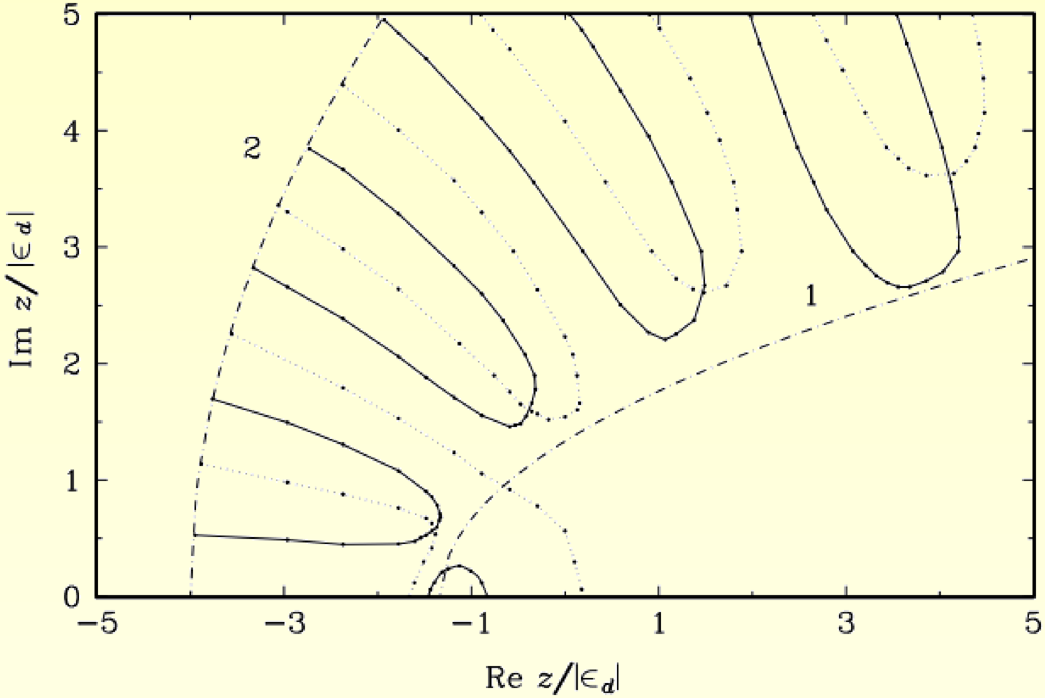


Fig. 1. Root locus curves of the real and imaginary parts of the scattering matrix $S_0(z)$ in case of helium trimer. The solid lines correspond to $\mathrm{Re}\, S_0(z) = 0$, while the tiny dashed lines, to $\mathrm{Im}\, S_0(z) = 0$. The Numbers 1, 2 denote the boundaries of the domains $\Pi^{(S)}$ and $\Pi^{(\Psi)}$, respectively. Complex roots of the function $S_0(z)$ are represented by the crossing points of the curves $\mathrm{Re}\, S_0(z) = 0$ and $\mathrm{Im}\, S_0(z) = 0$ and are located at $(-2.34 + i\,0.96)$ mK, $(-0.59 + i\,2.67)$ mK, $(2.51 + i\,4.34)$ mK and $(6.92 + i\,6.10)$ mK.

A.K.Motovilov, *Math.Nachrichten* **187**,147(1997)

E.A.K, A.K.Motovilov *Phys.At.Nucl.* **60**, 235 (1997)

E.A.K., A.K.Motovilov,Y.K.Ho *Nucl.Phys.A* **684**, 623 (2001)

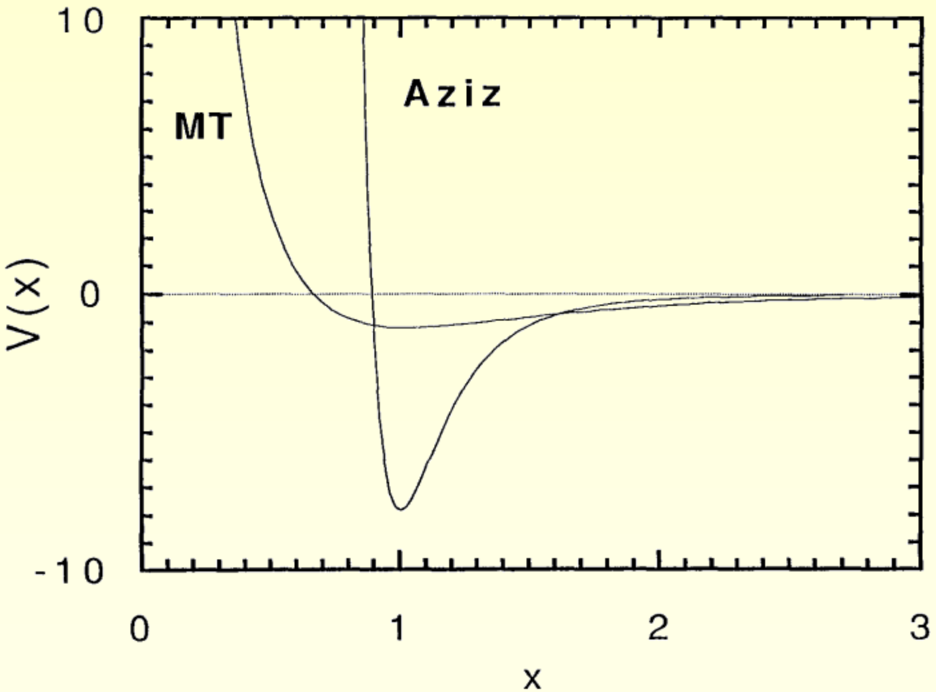
^3H - system

Three-body, theory

MT I-III

E_d = - 2.224 MeV

Virtual state of (nnp) -2.69 MeV (0.47 MeV)



A=3
Nucl. Phys. A **848**, 1 (2010)

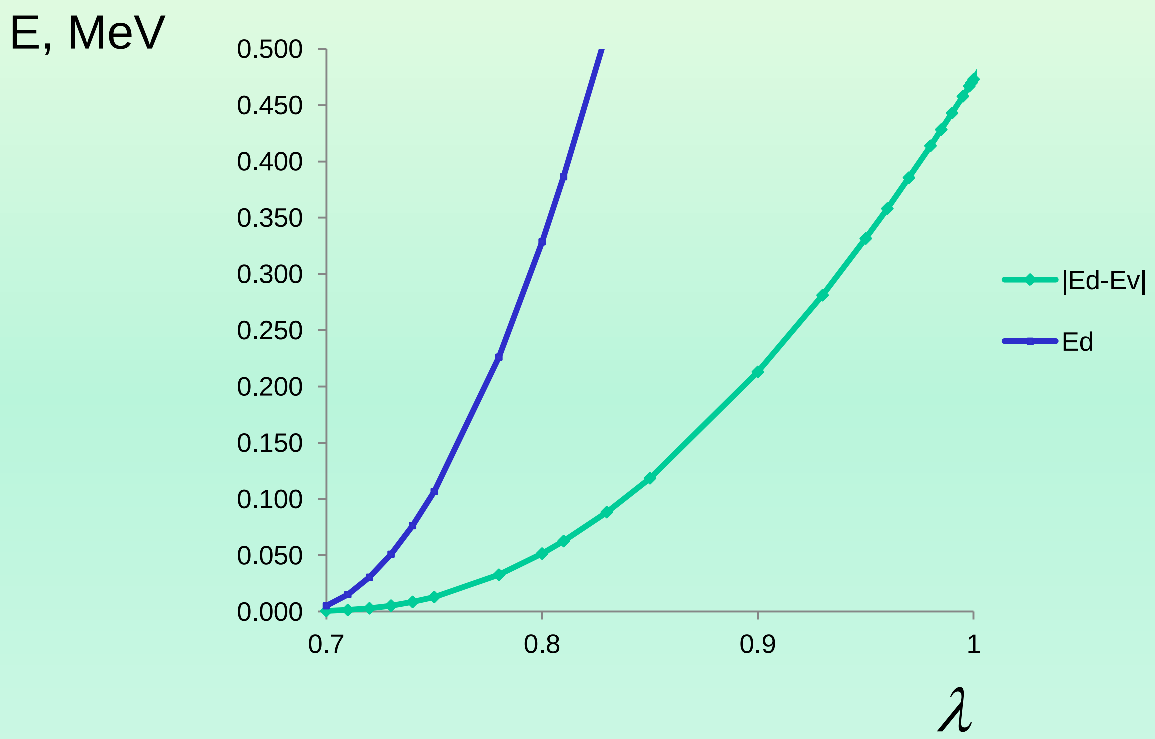
0.48 MeV

Orlov, Nikitina *Yad.Fiz.* 69 (2006)

Fig. 1. The dimensionless MT I-III $^3S_1$ nuclear and the HFD-B atomic potentials. The length unit for each potential is the position of its minimum (L). The potentials are made dimensionless according to the Schrödinger equation, i.e. $v = L^2 V/(\hbar^2/m)$

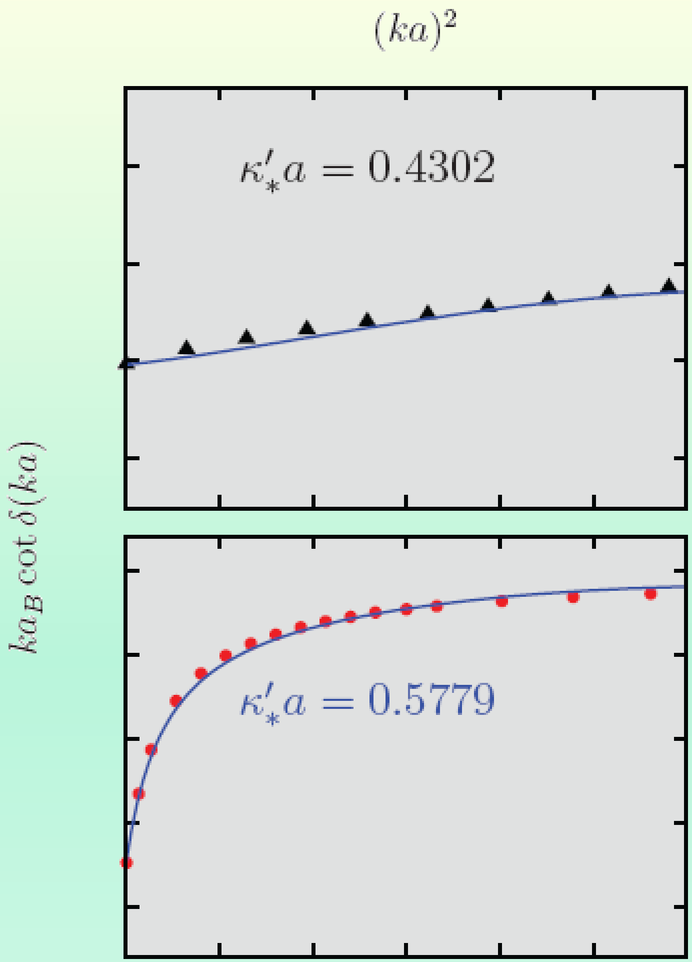
Carbonell, Gignoux, Merkuriev *FBS* **15**, 15 (1993)

$$V(x) = \lambda V_{MT}(x)$$



$$V(x) = \lambda V_c(x)$$

A.Kievsky, M.Gattobigio,
Phys. Rev. A **87**, 052719 (2013)



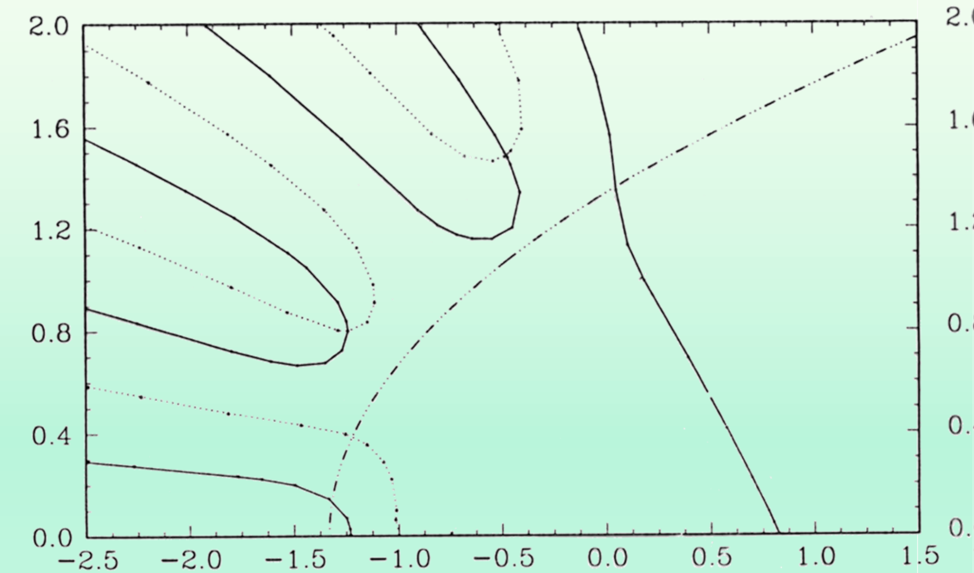
$^4He - {}^4He_2$ scattering

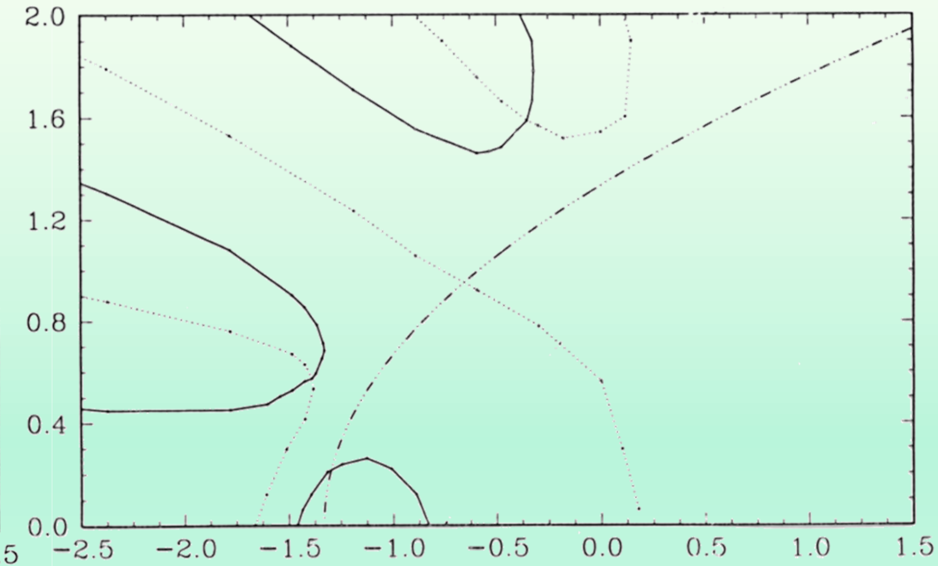
n – d scattering
in the doublet channel

$$ka_B \cot \delta = c_1(ka) + c_2(ka) \cot[s_0 \ln(\kappa'_* a) + \phi(ka)] \, .$$

Root locus curve of scattering matrix

$\mathrm{Im}\, z / |\varepsilon_d|$





^{3}H

$\mathrm{Re}\, z / |\varepsilon_d|$

^{4}He$_3$

S-matrix root lines in nnp system

S-matrix root lines in ^{4}He$_3$ system

Solid line - Re(S)=0, tiny dashed line – Im(S)=0

Conclusion

- We employed formalism which is suitable for three-body atomic systems interacted via hard-core potential. This method let us calculate bound states and scattering observables.
- It was demonstrated how the Efimov states emerge from the virtual ones when decreasing the strength of the interaction.
- It was shown similar properties of a very different systems: helium trimer and nnd system

Thank you!

$^4He_2$

Two-body, theory

- Potential models: Aziz et al. – HFD-B (1987), LM2M2 (1991),Tang et al. – TTY (1995)

$$V_{\mathrm{HFD-B}}(x) = \varepsilon \left\{ A \exp(-\alpha\zeta + \beta\zeta^2) - \left[\frac{C_6}{\zeta^6} + \frac{C_8}{\zeta^8} + \frac{C_{10}}{\zeta^{10}} \right] F(\zeta) \right\}$$

where $\zeta = x/r_m.$ and $F(\zeta) = \begin{cases} \exp\left[-\left(D/\zeta - 1\right)\right]^2, & \text{if } \zeta \le D \\ 1, & \text{if } \zeta > D\,. \end{cases}$